A Preorder Relation for Markov Reward Processes

David Daly [a]

[a] *IBM T.J. Watson Research Center*
Yorktown Heights, NY, 10958, U.S.A.
dmdaly@us.ibm.com

Peter Buchholz [b]

[b] *Informatik IV, Universität Dortmund*
D-44221 Dortmund, Germany
peter.buchholz@udo.edu

William H. Sanders [c]

[c] *Coordinated Science Laboratory*
University of Illinois at Urbana-Champaign
1308 W. Main St., Urbana, IL, U.S.A.
whs@crhc.uiuc.edu

Abstract

A new preorder relation is introduced that orders states of a Markov process with an additional reward structure according to the reward gained over any interval of finite or infinite length. The relation allows the comparison of different Markov processes and includes as special cases monotone and lumpable Markov processes.

Key words: Discrete Time Markov Chains; Rewards; Stochastic Orders; Comparison of Markov Processes; Aggregation

⋆ This material is based upon work supported by the National Science Foundation under Grant No. CCR-0311616, by Pioneer Hi-Bred International, Inc., and by the Deutsche Forschungsgemeinschaft (DFG) in the SFB 559.

1 Introduction and Basic Definitions

We consider Markov processes X_k on some finite or countable state space $\mathcal{S}$, an initial distribution p_0 ($p_0 : \mathcal{S} \to \mathbb{R}_+$ and $\sum_{i\in\mathcal{S}} p_0(i) = 1.0$) and an additional reward function r ($r : \mathcal{S} \to \mathbb{R}_+$) which assigns a non-negative reward to the states of the Markov process. Throughout the paper we introduce our approach for Markov processes in discrete time (DTMCs). However, as shown below, a continuous time Markov process (CTMC) with bounded transitions rates can be transformed into a DTMC via uniformization [10] and most of the presented results hold as well for uniformizable CTMCs. Let $\mathbf{P}$ be the transition matrix of the DTMC. $\mathbf{P}$ is a stochastic matrix of dimension $|\mathcal{S}|$ where $|\mathcal{S}|$ is the size of the state space and $\mathbf{P}(x, y)$ is the transition probability from state x to y. Similarly, the initial distribution and the reward function can be described as a $|\mathcal{S}|$-dimensional row vector $\mathbf{p}_0$ and a $|\mathcal{S}|$-dimensional column vector $\mathbf{r}$, respectively. We use the notation $DM = (\mathcal{S}, \mathbf{P}, \mathbf{p}_0, \mathbf{r})$ for a DTMC with rewards.

The state distribution of the DTMC after k steps is given by

$$\mathbf{p}_k = \mathbf{p}_0\mathbf{P}^k \ . \tag{1}$$

The reward $R_k(\mathbf{p}_0)$ after k steps is a random variable with distribution function

$$F_{R_k(\mathbf{p}_0)}(s) = \sum_{x\in\mathcal{S},\mathbf{r}(x)\leq s} \mathbf{p}_k(x) \text{ for all } s \in \mathbb{R}_+ \tag{2}$$

and expectation

$$E[R_k(\mathbf{p}_0)] = \sum_{x\in\mathcal{S}} \mathbf{p}_k(x)\mathbf{r}(x) = \mathbf{p}_0\mathbf{P}^k\mathbf{r} \ . \tag{3}$$

For the computation of $F_{R_k(\mathbf{p}_0)}(s)$ let $\nu_1, \ldots, \nu_m$ be the sequence of distinct rewards in $\mathbf{r}$ such that $\nu_i < \nu_{i+1}$ ($i = 1, \ldots, m-1$), for each $x \in \mathcal{S}$ exists exactly one ν_l with $\mathbf{r}(x) = \nu_l$ and for each ν_l exists at least one $x \in \mathcal{S}$ such that $\nu_l = \mathbf{r}(x)$. Define a matrix $\mathbf{R} \in \{0, 1\}^{n\times m}$ with $\mathbf{R}(x, l) = 1$ if $\mathbf{r}(x) = \nu_l$ and 0 otherwise. If $\mathbf{p}$ is a distribution over $\mathcal{S}$, then $\mathbf{pR}$ is the corresponding reward distribution. This implies

$$F_{R_k(\mathbf{p}_0)}(s) = \sum_{l:\nu_l\leq s} \left(\mathbf{p}_0\mathbf{P}^k\mathbf{R}\right)(l) \tag{4}$$

where $(\mathbf{pP})(x)$ denotes the xth element of vector $\mathbf{pP}$. For notational convenience we can identify the random variable $R_k(\mathbf{p}_0)$ with its distribution

$\mathbf{p}_0\mathbf{P}^k\mathbf{R}$.

For $0 \leq k_1 \leq k_2$ $AR_{k_1,k_2}(\mathbf{p}_0)$ is the accumulated reward with expectation

$$E[AR_{k_1,k_2}(\mathbf{p}_0)] = \sum_{k=k_1}^{k_2} R_k(\mathbf{p}_0) = \mathbf{p}_{k_1}\left(\sum_{k=0}^{k_2-k_1}\mathbf{P}^k\right)\mathbf{r}\ . \tag{5}$$

Following [8], a random variable X is stochastically larger than a random variable Y, denoted as $X \geq_{st} Y$, if $F_X(t) \leq F_Y(t)$ for all values of t. Obviously $X \geq_{st} Y$ implies $E[X] \geq E[Y]$. To apply the stochastic ordering for states of a DTMC, we define $R_k(\mathbf{p})$ as the reward gained in the kth step according to (2) when the initial distribution equals $\mathbf{p}$ (i.e., $\mathbf{p}_0 = \mathbf{p}$). Let $\mathbf{e}_x$ be a vector of appropriate length (i.e., the length equals the number of states of the corresponding DTMC) where element x equals 1 and all other elements are equal to 0. If we consider different DTMCs, then we number them consecutively and denote by $DM^{(i)} = (\mathcal{S}^{(i)}, \mathbf{P}^{(i)}, \mathbf{p}_0^{(i)}, \mathbf{r}^{(i)})$ the process with number i. Similarly, $R_k^{(i)}(\mathbf{p}_0)$ and $AR_{k_1,k_2}^{(i)}(\mathbf{p}_0)$ are defined.

Definition 1.1 *Let $DM = (\mathcal{S}, \mathbf{P}, \mathbf{p}_0, \mathbf{r})$ be a DTMC with rewards and $x, y \in \mathcal{S}$. We say $x \geq_{st} y$ iff $R_k(\mathbf{e}_x) \geq_{st} R_k(\mathbf{e}_y)$ for all $k = 0, 1, \ldots$.*

Obviously the following relation holds.

$$R_k(\mathbf{e}_x) \geq_{st} R_k(\mathbf{e}_y) \ \Leftrightarrow\ \sum_{z=l}^{m}\left(\mathbf{e}_x\mathbf{P}^k\mathbf{R}\right)(z) \geq \sum_{z=l}^{m}\left(\mathbf{e}_y\mathbf{P}^k\mathbf{R}\right)(z) \tag{6}$$

for all $l \in \{0, \ldots, m\}$.

Definition 1.2 *Let $DM^{(i)} = (\mathcal{S}^{(i)}, \mathbf{P}^{(i)}, \mathbf{p}_0^{(i)}, \mathbf{r}^{(i)})$ $(i = 1, 2)$ be two DTMCs. We say $DM^{(1)} \geq_{st} DM^{(2)}$ iff $R_k^{(1)}(\mathbf{p}_0^{(1)}) \geq_{st} R_k^{(2)}(\mathbf{p}_0^{(2)})$ for all $k = 0, 1, \ldots$.*

Since $X_k \geq_{st} Y_k$ for $k \in \mathcal{K}$, where $\mathcal{K}$ is some set $\mathcal{K} \subseteq \mathbb{N}$, for mutually independent X_k and mutually independent Y_k, implies

$$\sum_{k\in\mathcal{K}} \alpha_k X_k \geq_{st} \sum_{k\in\mathcal{K}} \alpha_k Y_k \tag{7}$$

for $\alpha_k \geq 0$ [8], we have

$$x \geq_{st} y \ \Rightarrow \begin{cases} E[R_k(\mathbf{e}_x)] \geq E[R_k(\mathbf{e}_y)] & \\ AR_{k_1,k_2}(\mathbf{e}_x) \geq_{st} AR_{k_1,k_2}(\mathbf{e}_y) & \text{for all } 0 \leq k_1 \leq k_2 \\ E[AR_{k_1,k_2}(\mathbf{e}_x)] \geq E[AR_{k_1,k_2}(\mathbf{e}_y)] & \text{for all } 0 \leq k_1 \leq k_2 \end{cases} \tag{8}$$

and similarly

$$DM^{(1)} \geq_{st} DM^{(2)} \;\Rightarrow \tag{9}$$
$$\begin{cases} E[R_k^{(1)}(\mathbf{p}_0^{(1)})] \geq E[R_k^{(2)}(\mathbf{p}_0^{(2)})] & \\ AR_{k_1,k_2}^{(1)}(\mathbf{p}_0^{(1)}) \geq_{st} AR_{k_1,k_2}^{(2)}(\mathbf{p}_0^{(2)}) & \text{for all } 0 \leq k_1 \leq k_2 \\ E[AR_{k_1,k_2}^{(1)}(\mathbf{p}_0^{(1)})] \geq E[AR_{k_1,k_2}^{(2)}(\mathbf{p}_0^{(2)})] & \text{for all } 0 \leq k_1 \leq k_2 \end{cases}$$

We now extend the approach to CTMCs. Let $CM = (\mathcal{S}, \mathbf{Q}, \mathbf{p}_0, \mathbf{r})$ be a CTMC with generator matrix $\mathbf{Q}$ where $\mathbf{Q}(i,j)$ is the transition rate between states i and j for $i \neq j$ and $\mathbf{Q}(i,i) = -\sum_{j\in\mathcal{S},j\neq i} \mathbf{Q}(i,j)$. We assume that $|\mathbf{Q}(i,j)| \leq \alpha < \infty$ for some constant α. The uniformized DTMC DM for CTMC CM has the same state space, the same initial distribution, the same reward values and a transition matrix

$$\mathbf{P} = \mathbf{Q}/\alpha + \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix of appropriate dimension. Let $\mathbf{p}_t$ be the distribution of a CTMC at time t, then the distribution function of the reward $R_t(\mathbf{p}_0)$ at time t is defined using (2) with $\mathbf{p}_t$ instead of $\mathbf{p}_k$. Similarly, the accumulated reward in the interval $[t_1, t_2]$ with $t_1 \leq t_2$ is given by

$$AR_{t_1,t_2}(\mathbf{p}_0) = \int_{t=t_1}^{t_2} R_t(\mathbf{p}_0)dt \; . \tag{10}$$

It is well known that the distribution $\mathbf{p}_t$ of a CTMC and the distributions $\mathbf{p}_k$ ($k = 0, 1, \ldots$) of the corresponding uniformized DTMC are related as follows [10].

$$\mathbf{p}_t = e^{-\alpha t} \sum_{k=0}^{\infty} \mathbf{p}_k \frac{(\alpha t)^k}{k!} \tag{11}$$

Order relations for CTMCs can be derived via the corresponding uniformized DTMC as shown in the following definition and theorems.

Definition 1.3 *Let $CM = (\mathcal{S}, \mathbf{Q}, \mathbf{p}_0, \mathbf{r})$ be a CTMC with rewards and $x, y \in \mathcal{S}$. We say $x \geq_{st} y$ iff $R_t(\mathbf{e}_x) \geq_{st} R_t(\mathbf{e}_y)$ for all $t \geq 0$.*

Let $CM^{(i)} = (\mathcal{S}^{(i)}, \mathbf{Q}^{(i)}, \mathbf{p}_0^{(i)}, \mathbf{r}^{(i)})$ ($i = 1, 2$) be two CTMCs. We say $CM^{(1)} \geq_{st} CM^{(2)}$ iff $R_t^{(1)}(\mathbf{p}_0) \geq_{st} R_t^{(2)}(\mathbf{p}_0)$ for all $t \geq 0$.

Theorem 1.1 *Let $CM = (\mathcal{S}, \mathbf{Q}, \mathbf{p}_0, \mathbf{r})$ be a uniformizable CTMC and $DM_\alpha = (\mathcal{S}, \mathbf{Q}/\alpha + \mathbf{I}, \mathbf{p}_0, \mathbf{r})$ be the uniformized DTMC for uniformization rate α. If for some α with $\max_{i \in \mathcal{S}} |\mathbf{Q}(i,i)| \leq \alpha < \infty$: $x \geq_{st} y$ in DM_α, then $x \geq_{st} y$ in CM.*

Proof. Let $R_t(\mathbf{e}_x)$ be the reward at time t when starting in state x at time 0. We have to show $R_t(\mathbf{e}_x) \geq_{st} R_t(\mathbf{e}_y)$. Since $x \geq_{st} y$ in DM_α we know $R'_k(\mathbf{e}_x) = \mathbf{e}_x \mathbf{P}^k \mathbf{R} \geq_{st} \mathbf{e}_y \mathbf{P}^k \mathbf{R} = R'_k(\mathbf{e}_y)$ for all $k = 1, 2, \ldots$. By (7) the following relation holds.

$$R_t(\mathbf{e}_x) = e^{-\alpha t} \sum_{k=0}^{\infty} \frac{(\alpha t)^k}{k!} \mathbf{e}_x \mathbf{P}^k \mathbf{R} \geq_{st} e^{-\alpha t} \sum_{k=0}^{\infty} \frac{(\alpha t)^k}{k!} \mathbf{e}_y \mathbf{P}^k \mathbf{R} = R_t(\mathbf{e}_y)$$

which implies that x is stochastically larger than y in CM. □

It is interesting to note that the previous theorem includes no equivalence since the relation depends on the choice of α which implies that not all uniformized discrete time chains preserve the stochastic ordering of the CTMC. The following theorem shows that it is sufficient to choose α large enough.

Theorem 1.2 *Let $CM = (\mathcal{S}, \mathbf{Q}, \mathbf{p}_0, \mathbf{r})$ be a uniformizable CTMC and $DM_\alpha = (\mathcal{S}, \mathbf{Q}/\alpha + \mathbf{I}, \mathbf{p}_0, \mathbf{r})$, $DM_\beta = (\mathcal{S}, \mathbf{Q}/\alpha + \mathbf{I}, \mathbf{p}_0, \mathbf{r})$ be two uniformized DTMCs with uniformization rates $\alpha \leq \beta$ ($\max_{i \in \mathcal{S}}(|\mathbf{Q}(i,i)| \leq \alpha < \infty$). If $x \geq_{st} y$ in DM_α, then $x \geq_{st} y$ in DM_β.*

Proof. Let $\rho = \alpha/\beta$, then $\mathbf{P}_\beta = \rho \mathbf{P}_\alpha + (1-\rho)\mathbf{I}$. We have to show that if $\mathbf{e}_x \mathbf{P}_\alpha^k \mathbf{R} \geq_{st} \mathbf{e}_y \mathbf{P}_\alpha^k \mathbf{R}$ for all $k = 0, 1, \ldots$, then $\mathbf{e}_x \mathbf{P}_\beta^k \mathbf{R} \geq_{st} \mathbf{e}_y \mathbf{P}_\beta^k \mathbf{R}$. Consider for some fixed k the following relation

$$\begin{array}{lcl} \mathbf{e}_x \mathbf{P}_\beta^k \mathbf{R} & = & \mathbf{e}_x (\rho \mathbf{P}_\alpha + (1-\rho)\mathbf{I})^k \mathbf{R} \quad = \\ \sum_{l=0}^{k} \binom{l}{k} \rho^l (1-\rho)^{k-l} \mathbf{e}_x \mathbf{P}_\alpha^l \mathbf{r} & \geq_{st} & \sum_{l=0}^{k} \binom{l}{k} \rho^l (1-\rho)^{k-l} \mathbf{e}_y \mathbf{P}_\alpha^l \mathbf{R} = \\ \mathbf{e}_y (\rho \mathbf{P}_\alpha + (1-\rho)\mathbf{I})^k \mathbf{R} & = & \mathbf{e}_y \mathbf{P}_\beta^k \mathbf{R} \end{array}$$

which holds by (7) and proves the theorem. □

The next theorem relates different CTMCs via their uniformized DTMCs.

Theorem 1.3 *Let $CM^{(i)} = (\mathcal{S}^{(i)}, \mathbf{Q}^{(i)}, \mathbf{p}_0^{(i)}, \mathbf{r}^{(i)})$ ($i = 1, 2$) be two CTMCs and $DM^{(i)} = (\mathcal{S}^{(i)}, \mathbf{Q}^{(i)}/\alpha + \mathbf{I}, \mathbf{p}_0^{(i)}, \mathbf{r}^{(i)})$ the corresponding uniformized DTMCs for some appropriate uniformization rate α with $\max_{i=1,2}(\max_{x \in \mathcal{S}^{(i)}}(\mathbf{Q}^{(i)}(x,x)|)) \leq \alpha < \infty$. If $DM^{(1)} \geq_{st} DM^{(2)}$ then $CM^{(1)} \geq_{st} CM^{(2)}$.*

Proof. The proof follows from the proof of theorem 1.2. □

Due to (11) it follows that for the stochastic ordering of states in a CTMC or complete CTMCs, the relations (8) and (9) hold as well. The above relation between DTMCs and CTMCs shows that it is often sufficient to consider DTMCs for stochastic orders.

Stochastic orders among states of a DTMC or different DTMCs are useful from a theoretical and practical viewpoint since they allow the comparison of processes and the computation of bounding processes. However, the general computation of the relation $\geq_{st}$ is cumbersome. Therefore we present in the following section a weaker order among states and DTMCs that implies $\geq_{st}$ and has a finite characterization for finite state spaces. Afterwards, the generation of bounding DTMCs for a given DTMC and finally the relation between the new order and the well known concepts of monotonicity [3,6,8] and lumpability [1,7] is investigated, followed by a brief example.

2 A Preorder Relation for Markov Reward Processes

Before we present a new preorder relation and some of its features, some notation is introduced. $\mathbf{P}(x\bullet)$ denotes row x of matrix $\mathbf{P}$. A preorder $\succeq$ is a reflexive and transitive relation among the states from $\mathcal{S}$. We use the notation $x \succeq y$ to indicate that x and y are related according to preorder $\succeq$.

Definition 2.1 *A decomposition ϕ of a non-negative vector $\mathbf{a} \in \mathbb{R}_{\geq 0}^{|\mathcal{S}|}$ over finite or countable state space $\mathcal{S}$ is a set of vectors $(\mathbf{a}_1, \ldots, \mathbf{a}_{I_\phi})$ with $\mathbf{a}_i \in \mathbb{R}_{\geq 0}^{|\mathcal{S}|}$ and $\sum_{i=1}^{I_\phi} \mathbf{a}_i = \mathbf{a}$. I_ϕ is the size of the decomposition, which can be infinite.*

In a decomposition ϕ a non-negative vector $\mathbf{a}$ of length $|\mathcal{S}|$ is decomposed into a set of I_ϕ non-negative vectors $\mathbf{a}_i$ such that for every element $x \in \mathcal{S}$ $\mathbf{a}(x) = \sum_{i=1}^{I_\phi} \mathbf{a}_i(x)$. Next we extend the definition of a preorder $\succeq$ on the states of a model to define a preorder on probability vectors.

Definition 2.2 *Let $\mathcal{S}$ be a state space, $\succeq$ be a preorder on the states of $\mathcal{S}$, and $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{\geq 0}^{|\mathcal{S}|}$, then $\mathbf{a} \succeq \mathbf{b}$ iff a decomposition $\phi = (\mathbf{a}_1, \ldots, \mathbf{a}_{|\mathcal{S}|})$ of size $|\mathcal{S}|$ exists such that for every $x \in \mathcal{S}$: $\sum_{y \succeq x} \mathbf{a}_x(y) \geq \mathbf{b}(x)$ and $\mathbf{a}_x(y) = 0$ for $\neg(y \succeq x)$.*

The definition implies that $\mathbf{a}_x$ contains at least one non-zero element if $\mathbf{b}(x) > 0$ and it may contain up to $|\{y | y \succeq x\}|$ non-zero elements. The previous definition can be used to define preorder relations that order states according to their contribution to the reward measures.

Definition 2.3 *A preorder $\succeq$ on the state space of a DTMC $DM = (\mathcal{S}, \mathbf{P}, \mathbf{p}_0, \mathbf{r})$*

is reward preserving, iff $x \succeq y$ implies

(1) $\mathbf{r}(x) \geq \mathbf{r}(y)$ *and*
(2) $\mathbf{P}(x\bullet) \succeq \mathbf{P}(y\bullet)$.

Theorem 2.1 *If for two states $x, y \in \mathcal{S}$ $x \succeq y$ and $\succeq$ is reward preserving, then $x \geq_{st} y$.*

Proof. We have to show that $F_{R_k(\mathbf{e}_x)}(z) \leq F_{R_k(\mathbf{e}_y)}(z)$ for all $k \in \mathcal{S}$ and $z \in \mathbb{R}_+$. The proof is done by induction over k. For $k = 0$ the result follows since $\mathbf{r}(x) \geq \mathbf{r}(y)$.

Now assume that the result has been proved for k, we show that it also holds for $k+1$. We have

$$\mathbf{e}_x \mathbf{P}^{k+1} \mathbf{R} = \mathbf{P}(x\bullet)\mathbf{P}^k \mathbf{R} \text{ and } \mathbf{e}_y \mathbf{P}^{k+1} \mathbf{R} = \mathbf{P}(y\bullet)\mathbf{P}^k \mathbf{R} \ .$$

Since $\mathbf{P}(x\bullet) \succeq \mathbf{P}(y\bullet)$ a decomposition $\phi = (\mathbf{p}_1, \ldots, \mathbf{p}_{|\mathcal{S}|})$ of $\mathbf{P}(x\bullet)$ exists such that for each $v \in \mathcal{S}$: $\sum_{u \succeq v} \mathbf{p}_v(u) \geq \mathbf{P}(y, v)$. Thus, we have for every $\mathbf{P}(y, v) > 0$ a vector $\mathbf{p}_v$ and it follows by the induction assumption that

$$\mathbf{p}_v \mathbf{P}^k \mathbf{R} \geq_{st} \mathbf{P}(y, v)\mathbf{e}_v \mathbf{P}^k \mathbf{R}$$

and also

$$\sum_{v \in \mathcal{S}} \mathbf{p}_v \mathbf{P}^k \mathbf{R} \geq_{st} \sum_{v \in \mathcal{S}} \mathbf{P}(y, v)\mathbf{e}_v \mathbf{P}^k \mathbf{R} \Rightarrow \mathbf{P}(x\bullet)\mathbf{P}^k \mathbf{R} \geq_{st} \mathbf{P}(y\bullet)\mathbf{P}^k \mathbf{R}$$

□

To extend relation $\succeq$ to DTMCs instead of states of a single DTMC, we first define the union of DTMCs.

Definition 2.4 *Let $DM^{(i)} = (\mathcal{S}^{(i)}, \mathbf{P}^{(i)}, \mathbf{p}_0^{(i)}, \mathbf{r}^{(i)})$ $(i = 1, 2)$ with $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \emptyset$, then $DM^{(12)} = DM^{(1)} \cup DM^{(2)} = (\mathcal{S}^{(12)}, \mathbf{P}^{(12)}, \mathbf{p}_0^{(12)}, \mathbf{r}^{(12)})$ is defined as $\mathcal{S}^{(12)} = \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)}$,*

$$\mathbf{P}^{(12)} = \begin{pmatrix} \mathbf{P}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{(2)} \end{pmatrix}, \ \mathbf{p}_0^{(12)} = 0.5 \left(\mathbf{p}_0^{(1)}, \mathbf{p}_0^{(2)}\right) \ \textit{and} \ \ \mathbf{r}^{(12)} = \left(\mathbf{r}^{(1)}, \mathbf{r}^{(2)}\right)$$

Theorem 2.2 *Let $DM^{(i)} = (\mathcal{S}^{(i)}, \mathbf{P}^{(i)}, \mathbf{p}_0^{(i)}, \mathbf{r}^{(i)})$ $(i = 1, 2)$ and $\succeq$ be a reward preserving preorder on the union of both processes, then $DM^{(1)} \geq_{st} DM^{(2)}$ if $\left(\mathbf{p}_0^{(1)}, \mathbf{0}\right) \succeq \left(\mathbf{0}, \mathbf{p}_0^{(2)}\right)$.*

Proof. By assumption a decomposition $(\mathbf{0},\ldots,\mathbf{0},\mathbf{a}_1,\ldots,\mathbf{a}_{|\mathcal{S}_2|})$ of $\mathbf{p}_0^{(1)}$ exists such that for all $x \in \mathcal{S}_2$ $\sum_{y\succeq x}\mathbf{a}_x(y) \geq \mathbf{p}_0^{(2)}(x)$. Observe that all non-zero elements of $\mathbf{a}_x$ belong to states from $\mathcal{S}^{(2)}$. Since $\succeq$ is reward preserving, $x \succeq y$ implies $R_k^{(12)}(\mathbf{e}_x) \geq_{st} R_k^{(12)}(\mathbf{e}_y)$ and $R_k^{(12)}(\mathbf{e}_z) = R_k^{(i)}(\mathbf{e}_z)$ [1] for $z \in \mathcal{S}^{(i)}$. This implies that $\sum_{x\in\mathcal{S}^{(2)}}\sum_{y\succeq x}\mathbf{a}_x(y)R_k^{(12)}(\mathbf{e}_y) \geq_{st} \sum_{x\in\mathcal{S}^{(2)}}\mathbf{p}_0^{(2)}(x)R_k^{(12)}(\mathbf{e}_x)$ for all $k = 0, 1, \ldots$. □

For an algorithmic generation of reward preserving preorders define a sequence $\succeq_k$ $(k = 0, 1, \ldots)$ of relations for a DM as follows

(1) $x \succeq_0 y \Leftrightarrow \mathbf{r}(x) \geq \mathbf{r}(y)$,
(2) $x \succeq_k y$ for $k > 0$ iff
  (a) $x \succeq_{k-1} y$ and
  (b) $\mathbf{P}(x\bullet) \succeq_{k-1} \mathbf{P}(y\bullet)$.

Due to condition 2a, $\succeq_{k+1}$ contains no more relations between states than $\succeq_k$. We say $\succeq_{k+1}$ is finer than $\succeq_k$. The following theorem shows that $\succeq_k$ is indeed a family of preorders.

Theorem 2.3 *All relations $\succeq_k$ are transitive and reflexive.*

Proof. The proof of reflexivity is trivial. For $k = 0$ transitivity holds since $x \succeq_0 y$ and $y \succeq_0 z$ implies $\mathbf{r}(x) \geq \mathbf{r}(y) \geq \mathbf{r}(z)$ which implies $x \succeq_0 z$. Now assume that transitivity has been proved for k we show that it then also holds for $k + 1$.

Since $x \succeq_{k+1} y$ a decomposition $(\mathbf{a}_1,\ldots,\mathbf{a}_{|\mathcal{S}|})$ of $\mathbf{P}(x\bullet)$ exists such that $\sum_{u\succeq_k v}\mathbf{a}_v(u) \geq \mathbf{P}(y,v)$ and similarly since $y \succeq_{k+1} z$ a decomposition $(\mathbf{b}_1,\ldots,\mathbf{b}_{|\mathcal{S}|})$ of $\mathbf{P}(y\bullet)$ exists such that $\sum_{u\succeq_k v}\mathbf{b}_v(u) \geq \mathbf{P}(z,v)$. We have to show that a decomposition $(\mathbf{c}_1,\ldots,\mathbf{c}_{|\mathcal{S}|})$ of $\mathbf{P}(x\bullet)$ exists such that $\sum_{w\succeq_k v}\mathbf{c}_v(w) \geq \mathbf{P}(z,v)$.

We construct $\mathbf{c}_v$ from $\mathbf{a}_.$ and $\mathbf{b}_v$ as follows

$$\mathbf{c}_v(w) = \sum_{u\succeq_k v}\frac{\mathbf{b}_v(u)}{\mathbf{P}(y,u)}\cdot\mathbf{a}_u(w)$$

where $w \succeq_k v$. Since $\succeq_k$ is transitive, the above construction generates valid vectors $\mathbf{c}$ such that also $\succeq_{k+1}$ is transitive. □

Theorem 2.4 *If $\succeq_k = \succeq_{k+1}$, then $\succeq_k = \succeq_{k+l}$ for all $l > 0$.*

[1] Observe that the vectors $\mathbf{e}_z$ are of different length depending on the size of the corresponding state space.

Proof. The proof is straightforward, since the basic relation remains unchanged and no further refinement takes place. □

If the sequence $\succeq_k$ reaches a fixed point where $\succeq_k = \succeq_{k+1}$, then we denote the fixed point as $\succeq_\sim$. Observe that for finite state spaces the fixed point is always reached after at most $|\mathcal{S}|$ iterations. The following theorem shows that $\succeq_\sim$ is indeed the coarsest reward preserving preorder which means that it contains the greatest number of relations between states among all reward preserving preorders.

Theorem 2.5 *If $\succeq_*$ is a reward preserving preorder on the state space of a DM where the fixed point $\succeq_\sim$ exists, then $x \succeq_* y \Rightarrow x \succeq_\sim y$.*

Proof. The proof is done inductively over the preorders $\succeq_k$. First notice that $x \succeq_0 y$ is necessary for $x \succeq_* y$ because otherwise $\succeq_*$ cannot be reward preserving. Now assume that we proved the assumption for k, i.e., $x \succeq_* y \Rightarrow x \succeq_k y$. We show that it also holds for $k+1$. Let x and y be two states such that $x \succeq_k y$ but $\neg(x \succ_{k+1} y)$. This implies that no appropriate decomposition of $\mathbf{P}(x\bullet)$ exists. Since $\succeq_*$ contains no more pairs of related states than $\succeq_k$ it is obvious that also no decomposition of $\mathbf{P}(x\bullet)$ exists according to $\succeq_*$ such that $\neg(x \succeq_* y)$ follows from $\neg(x \succeq_{k+1} y)$. Since $x \succeq_* y \Rightarrow x \succeq_k y$ holds for all k, it also holds for the fixed point $\succeq_\sim$ if it exists. □

Theorem 2.1 shows that $x \succeq_\sim y$ implies $x \geq_{st} y$. Now we show by means of an example that $x \geq_{st} y$ does not necessarily imply $x \succeq_\sim y$. The example contains 5 states and is characterized by the following transition matrix and reward vector.

$$\mathbf{P}\begin{pmatrix} 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}, \ \mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \ \text{yielding } \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We consider the states 1 and 5 and obtain

$$F_{R_1}(k) = \begin{cases} (1,0,0) & \text{for } k = 0 \\ (0, 0.5, 0.5) & \text{for } k = 1 \\ (0.5, 0.5, 0) & \text{for } k \geq 2 \end{cases} \quad F_{R_5}(k) = \begin{cases} (1,0,0) & \text{for } k = 0 \\ (0.5, 0.5, 0) & \text{for } k \geq 1 \end{cases}$$

which implies $1 \geq_{st} 5$. To represent $\succeq_k$ we use the notation $x \succeq_k \{y, z\}$ for $x \succeq_k y$ and $x \succeq_k z$. The following relations appear

$$\begin{array}{lll} 1 \succeq_0 \{1,5\} & 1 \succeq_1 \{1,5\} & 1 \succeq_2 \{1\} \\ 2 \succeq_0 \{1,2,3,4,5\} & 2 \succeq_1 \{2\} & 2 \succeq_2 \{2\} \\ 3 \succeq_0 \{1,3,4,5\} & 3 \succeq_1 \{3,4,5\} & 3 \succeq_2 \{3,4,5\} \\ 4 \succeq_0 \{1,3,4,5\} & 4 \succeq_1 \{4,5\} & 4 \succeq_2 \{4,5\} \\ 5 \succeq_0 \{1,5\} & 5 \succeq_1 \{5\} & 5 \succeq_2 \{5\} \end{array}$$

Such that $1 \succeq_2 5$ does not hold and therefor also $1 \succeq_\sim 5$ cannot hold. Since $x \succeq y$ for some reward preserving preorder $\succeq$ implies $x \succeq_\sim y$, the reverse direction of the implication in Theorem 2.1 is not true.

The computation of $\succeq_\sim$ is beyond the scope of this paper, but it is obvious that the relation can in principle be computed for finite state spaces. The central step is to check $\mathbf{a} \succeq \mathbf{b}$ for a given relation $\succeq$, which means to compute up to $|\mathcal{S}|^2$ elements $\mathbf{a}_x(y)$ such that

$$\mathbf{a}_x(y) \geq 0 \ , \ \sum_{x \in \mathcal{S}} \mathbf{a}_x(y) = \mathbf{a}(y) \text{ and } \sum_{y \succeq x} \mathbf{a}_x(y) \geq \mathbf{b}(x) \ .$$

In a practical realization several simplifications can be introduced to make the analysis feasible. For details we refer to [4].

3 Order Preserving Aggregation

The concept of stochastic orders of DTMCs allows the definition of bounding processes for a given DM. A bounding process yields lower or upper bounds for all reward measures of the original process and it is useful if it is somehow easier to analyze than the original DM. One way of building bounding processes is state aggregation which will be briefly presented.

Definition 3.1 *Let* $DM = (\mathcal{S}, \mathbf{P}, \mathbf{p}_0, \mathbf{r})$ *and* $\Pi = \{\mathcal{S}_1, \ldots, \mathcal{S}_J\}$ *be a partition of the state space.*

- Π *contains least states, iff for all* $\mathcal{S}_j$ *some* $x \in \mathcal{S}_j$ *exists such that* $x \preceq_\sim y$ *for all* $y \in \mathcal{S}_j$,
- Π *contains greatest states, iff for all* $\mathcal{S}_j$ *some* $x \in \mathcal{S}_j$ *exists such that* $x \succeq_\sim y$ *for all* $y \in \mathcal{S}_j$,

We consider only finite partitions such that J is some integer denoting the number of partition groups. Knowing $\succeq_\sim$ partitions with greatest and least states can be generated easily. The idea of aggregation is to generate a new DTMC where each group of states $\mathcal{S}_i$ is substituted by a single state which is greater/less than than all states in the block according to relation $\succeq_\sim$. The following definition describes one possible way to build these aggregated processes.

Definition 3.2 *Let $DM = (\mathcal{S}, \mathbf{P}, \mathbf{p}_0, \mathbf{r})$ be a DTMC with rewards and $\Pi = \{\mathcal{S}_1, \ldots, \mathcal{S}_J\}$ a partition of its state space. The following aggregated processes can be defined where $\tilde{\mathcal{S}} = \{1, \ldots, J\}$, $\tilde{\mathbf{P}} \in \mathbb{R}^{J \times J}$, $\tilde{\mathbf{p}}_0 \in \mathbb{R}^J$ and $\mathbf{r} \in \mathbb{R}^J$.*

(1) If Π contains least states and x_j^- is the least state of $\mathcal{S}_j$, then $DM^- = (\tilde{\mathcal{S}}, \tilde{\mathbf{P}}, \tilde{\mathbf{p}}_0, \tilde{\mathbf{r}})$ where $\tilde{\mathbf{p}}_0(j) = \sum_{y \in \mathcal{S}_j} \mathbf{p}_0(y)$, $\tilde{\mathbf{r}}(j) = \mathbf{r}(x_j^-)$ and $\tilde{\mathbf{P}}(j,l) = \sum_{y \in \mathcal{S}_l} \mathbf{P}(x_j^-, y)$.
(2) If Π contains greatest states and x_j^+ is the greatest state $\mathcal{S}_j$, then $DM^+ = (\tilde{\mathcal{S}}, \tilde{\mathbf{P}}, \tilde{\mathbf{p}}_0, \tilde{\mathbf{r}})$ where $\tilde{\mathbf{p}}_0(j) = \sum_{y \in \mathcal{S}_j} \mathbf{p}_0(y)$, $\tilde{\mathbf{r}}(j) = \mathbf{r}(x_j^+)$ and $\tilde{\mathbf{P}}(j,l) = \sum_{y \in \mathcal{S}_l} \mathbf{P}(x_j^+, y)$.

Theorem 3.1 *The relation $DM^+ \geq_{st} DM \geq_{st} DM^-$ holds.*

Proof. We show that $DM^+ \succeq_\sim DM$. By assumption $x_j^+ \succeq_\sim y$ for all $y \in \mathcal{S}_j$. Thus, we have to show that $j \succeq_\sim x_j^+$ holds for all $j \in \tilde{\mathcal{S}}$. Due to the transitivity of $\succeq_\sim$ this implies $j \succeq_\sim x$ for all $x \in \mathcal{S}_j$. Obviously $\tilde{\mathbf{r}}(j) \geq \mathbf{r}(x_j^+)$ holds. Thus, it remains to find a partition $(\mathbf{a}_1, \ldots, \mathbf{a}_{|\mathcal{S}|})$ of $\tilde{\mathbf{P}}(j\bullet)$ such that $\sum_{z \succeq_\sim y} \mathbf{a}_y(z) \geq \mathbf{P}(x_j^+, y)$. Let $y \in \mathcal{S}_l$, then we set $\mathbf{a}_y = \mathbf{P}(x_j^+, y)\mathbf{e}_l$ which observes the above condition since $x_l^+ \succeq_\sim x$ for all $x \in \mathcal{S}_l$. For the comparison of the processes we finally have to show $(\tilde{\mathbf{p}}_0, \mathbf{0}) \succeq_\sim (\mathbf{0}, \mathbf{p}_0)$. Since $\tilde{\mathbf{p}}_0(j) = \sum_{x \in \mathcal{S}_j} \mathbf{p}_0(x)$ and $j \succeq_\sim x$ for $x \in \mathcal{S}_j$, the relation holds.

The proof for $DM \geq_{st} DM^-$ is completely analogous. □

4 Relation to Monotonicity and Lumpability

The following definition of stochastic domination and monotone matrices can be found in [3,6,8]. We consider here the finite case, extension to infinite state spaces can be found in [3].

Definition 4.1 *Let $\mathbf{a}, \mathbf{b}$ be two distribution vectors and $\mathbf{P}$ be stochastic matrix on some finite state space $\mathcal{S}$, then*

- $\mathbf{a}$ *dominates* $\mathbf{b}$ *written as* $\mathbf{a} \succ \mathbf{b}$ *iff* $\sum_{y \geq x}^{|\mathcal{S}|} \mathbf{a}(y) \geq \sum_{y \geq x}^{|\mathcal{S}|} \mathbf{b}(y)$ *for all* $x \in \mathcal{S}$. $\mathbf{a}$ *dominates* $\mathbf{b}$ *strictly, if the strict inequality holds.*

- $\mathbf{P}$ *is a monotone matrix, iff* $\mathbf{P}(x\bullet) \succ \mathbf{P}(y\bullet)$ *for* $x \geq y$. $\mathbf{P}$ *is strictly monotone if strict dominance holds.*

For monotone matrices $\mathbf{P}$ and $\mathbf{a} \succ \mathbf{b}$ also $\mathbf{aP} \succ \mathbf{bP}$ such that also $\mathbf{aP}^k \succ \mathbf{bP}^k$ for all $k \geq 0$ [6]. In the context of Markov processes with rewards this implies $\mathbf{aP}^k\mathbf{r} \geq \mathbf{bP}^k\mathbf{r}$ and $\mathbf{aP}^k\mathbf{R} \geq_{st} \mathbf{bP}^k\mathbf{R}$ for all reward vectors where $x \geq y$ implies $\mathbf{r}(x) \geq \mathbf{r}(y)$. Via uniformization similar results can be derived for continuous time Markov processes [6]. The following theorem shows the relation between our preorder and stochastic dominance and monotonicity.

Theorem 4.1 *Let* $DM = (\mathcal{S}, \mathbf{P}, \mathbf{p}_0, \mathbf{r})$, *then the following implications hold*

(1) If $\mathbf{P}$ *is a monotone matrix and* $\mathbf{r}(x) \geq \mathbf{r}(y)$ *for all* $x \geq y \Rightarrow x \succeq_\sim y$ *for all* $x \geq y$.
(2) If $x \succeq_\sim y$ *for all* $x \geq y$ *and* $\neg(y \succeq_\sim x)$ *for* $y < x \Rightarrow \mathbf{P}$ *is strictly monotone and* $\mathbf{r}(x) \geq \mathbf{r}(y)$ *for all* $x > y$.

Proof. 1) Since $\mathbf{r}(x) \geq \mathbf{r}(y)$ for $x \geq y$, $x \succeq_0 y$ holds. Now we have to compute a partition $(\mathbf{a}_1, \ldots, \mathbf{a}_{|\mathcal{S}|})$ of $\mathbf{P}(x\bullet)$ such that $\sum_{u \geq v} \mathbf{a}_v(u) \geq \mathbf{P}(y, v)$. For $v = |\mathcal{S}|$ we set $\mathbf{a}_{|\mathcal{S}|} = \mathbf{P}(y, |\mathcal{S}|)\mathbf{e}_{|\mathcal{S}|}$ which is possible since $\mathbf{P}(y, |\mathcal{S}|) \leq \mathbf{P}(x, |\mathcal{S}|)$. For the remaining elements $v < |\mathcal{S}|$ the relation $\sum_{u=v}^{|\mathcal{S}|} \mathbf{P}(x, u) - \sum_{u=v+1}^{|\mathcal{S}|} \mathbf{P}(y, u) \geq \mathbf{P}(y, v)$ assure that we can find an appropriate vector $\mathbf{a}_v$.

2) $\mathbf{r}(x) \geq \mathbf{r}(y)$ for $x > y$ has to hold since $x \succeq_\sim y \;\Rightarrow\; x \succeq_0 y$. For $x > y$ a partition $(\mathbf{a}_1, \ldots, \mathbf{a}_{|\mathcal{S}|})$ of $\mathbf{P}(x\bullet)$ exists such that $\sum_{u \succeq_\sim v} \mathbf{a}_v(u) = \sum_{u=v}^{|\mathcal{S}|} \mathbf{a}_v(u) \geq \mathbf{P}(j, v)$. This implies

$$\sum_{u \succeq_\sim v} \mathbf{P}(x, u) = \sum_{u=v}^{|\mathcal{S}|} \mathbf{P}(x, u) \geq \sum_{w=v}^{|\mathcal{S}|} \sum_{u=w}^{|\mathcal{S}|} \mathbf{a}_w(u) \geq \sum_{u=v}^{|\mathcal{S}|} \mathbf{P}(y, u) = \sum_{u \succeq_\sim v} \mathbf{P}(x, u)$$

which shows that $\mathbf{P}$ is a strictly monotone matrix. □

Another concept in Markov chains is lumpability [7].

Definition 4.2 *A partition* $\Pi = \{\mathcal{S}_1, \ldots, \mathcal{S}_J\}$ *on the state space of a DTMC* $DM = (\mathcal{S}, \mathbf{P}, \mathbf{p}_0, \mathbf{r})$ *is lumpable, iff for all* $i, j \in \{1, \ldots, J\}$ *and all* $x, y \in \mathcal{S}_i$:

(1) $\mathbf{r}(x) = \mathbf{r}(y)$ *and*
(2) $\sum_{z \in \mathcal{S}_j} \mathbf{P}(x, z) = \sum_{z \in \mathcal{S}_j} \mathbf{P}(y, z)$.

The lumpable partition with the least number of partition groups can be computed for finite state spaces with a partition refinement algorithm [2,5] and will be denoted as $\Pi_\sim$.

Theorem 4.2 *Let $DM = (\mathcal{S}, \mathbf{P}, \mathbf{p}_0, \mathbf{r})$, then the following equivalence holds*

$$x \succeq_\sim y \wedge y \succeq_\sim x \;\Leftrightarrow\; x, y \in \mathcal{S}_j \text{ for some } \mathcal{S}_j \in \Pi_\sim.$$

Proof. $\Leftarrow$: For $x, y \in \mathcal{S}_j$ $\mathbf{r}(i) = \mathbf{r}(j)$ holds. Now define $y \succeq x$ if $x, y \in \mathcal{S}_j \in \Pi_\sim$, then it is straightforward to show that $\succeq$ is reward preserving and therefore $x \succeq y$ implies $x \succeq_\sim y$.

$\Rightarrow$: Define a partition $\Pi = \{\mathcal{S}_1, \ldots, \mathcal{S}_J\}$ such that $x, y \in \mathcal{S}_j$ iff $x \succeq_\sim y \wedge y \succeq_\sim x$. Due to the transitivity of the relation all states from $\mathcal{S}_j$ are related via $\succeq_\sim$. Obviously $\mathbf{r}(x) = \mathbf{r}(y)$ holds in this case. Now consider some $\mathcal{S}_j$ such that no $u \notin \mathcal{S}_j$ exist which is larger than some $v \in \mathcal{S}_j$ according to $\succeq_\sim$. Such a $\mathcal{S}_j$ has to exist since $\succeq_\sim$ is transitive. For two states $x, y \in \mathcal{S}_l$ $(l = 1, \ldots, J)$ a partition $(\mathbf{a}_1, \ldots, \mathbf{a}_{|\mathcal{S}|})$ of $\mathbf{P}(x\bullet)$ has to exist such that $\sum_{u \succeq_\sim v} \mathbf{a}_v(u) = \sum_{u \in \mathcal{S}_j} \mathbf{a}_v(u) \geq \mathbf{P}(y, v)$ and by a symmetric argument a partition $(\mathbf{b}_1, \ldots, \mathbf{b}_{|\mathcal{S}|})$ of $\mathbf{P}(y\bullet)$ exists such that $\sum_{u \succeq_\sim v} \mathbf{b}_v(u) = \sum_{u \in \mathcal{S}_j} \mathbf{b}_v(u) \geq \mathbf{P}(x, v)$. Thus we have

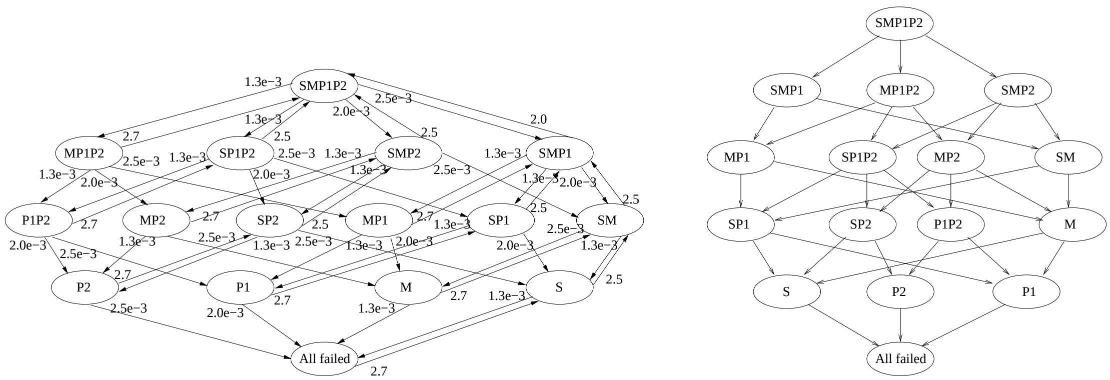
$$\sum_{v \in \mathcal{S}_j} \mathbf{P}(x, v) \geq \sum_{v \in \mathcal{S}_j} \sum_{u \in \mathcal{S}_j} \mathbf{a}_v(u) \geq \sum_{v \in \mathcal{S}_j} \mathbf{P}(y, v) \geq \sum_{v \in \mathcal{S}_j} \sum_{u \in \mathcal{S}_j} \mathbf{b}_v(u) \geq \sum_{v \in \mathcal{S}_j} \mathbf{P}(x, v)$$

which implies that $\geq$ can be substituted by $=$. Consequently, the argumentation can continue with some $\mathcal{S}_l$ such that no $u \notin \mathcal{S}_j \cup \mathcal{S}_l$ exists such that $u \succeq_\sim v$ for some $v \in \mathcal{S}_l$. This implies that Π is a lumpable partition. However since the first part of the proof shows that for $x, y \in \Pi_\sim$ $x \succeq_\sim y \wedge y \succeq_\sim x$ is necessary and $\Pi_\sim$ is the lumpable partition with the least number of groups $\Pi = \Pi_\sim$ has to hold. □

According to a lumpable partition, an aggregated DTMC $DM^\sim = (\mathcal{S}^\sim, \mathbf{P}^\sim, \mathbf{p}_0^\sim, \mathbf{r}^\sim)$ can be computed by substituting each partition group of states by a single state [1]. It is known that $F_{R_k(\mathbf{p}_0)}(x) = F_{R_k^\sim(\mathbf{p}_0^\sim)}(x)$ for all k and x [1,7], i.e. original and lumped process are indistinguishable. If $\succeq_\sim$ is computed on $DM^\sim$, then $x \succeq_\sim y$ implies $\neg(y \succeq_\sim x)$, the relation becomes antisymmetric and therefore $\succeq_\sim$ is a partial order in this case.

5 Example

We present a small example of the preorder and its application to build bounding aggregates. We consider the model of a processing node from a highly available distributed database presented in [9], and determine the preorder on the states of the processing node to create three bounding aggregates.



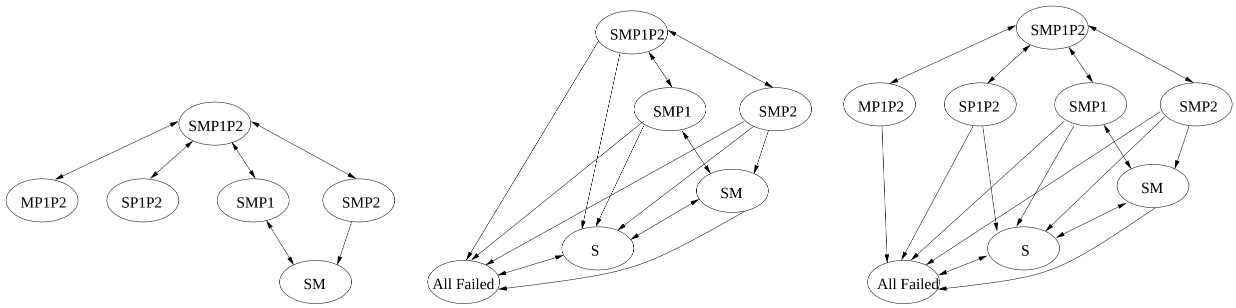
(a) Transition Diagram (b) Hasse Diagram

Fig. 1. Transition diagram and Hasse diagram of the partial order on the states of the processing node component for the distributed DB model.

The distributed database has multiple redundant components, and requires one component of each type to be operational for the database to be operational. One of the components of the database is a processing node consisting of two processors, a switch, and a memory. The processing node is considered operational if the switch, the memory, and at least one of the processors are operational. Each subcomponent fails independently with an exponential rate. Failed subcomponents are repaired one at a time, according to a priority scheme, with the switch repaired first, then the memory, then processor 1, and finally processor 2. Observe that due to the prioritized repair and the component specific failure rates of the processors, the model is not lumpable according to the processors.

We generated the preorder for the state space for one of the processing nodes to determine the node availability. The transition diagram for the processing node and the Hasse diagram of the partial order are shown in Figure 1. Each state is represented as a list of operational subcomponents, with an S for the switch, an M for the memory, P1 for processor 1, and P2 for processor 2. The processing node has the basic property that a failure always moves the component to a worse state, and a repair moves the component to a better state.

Based upon the preorder and the aggregation theory of Section 3, we developed a set of aggregates for the model. We reduce each processing subsystem to 6 states from the original 16 states by creating a partition of the states containing 6 subsets, each containing least and greatest states. Each processing subsystem has three operational states (SMP1P2, SMP1, SMP2), each of which we place into singleton sets in the partition. We partition the failed states into three sets based upon the highest priority failed component. Therefore, there is a set with the switch failed (All Failed, P1, P2, M, P1P2, MP1, MP2, MP1P2), the memory failed (S, SP1, SP2, SP1P2), and the processors failed (SM). The singleton sets trivially have greatest and least states, and



(a) 6 State Upper Bound (b) 6 State Lower Bound (c) 8 State Lower Bound

Fig. 2. Upper- and lower-bounding aggregates for one processing node.

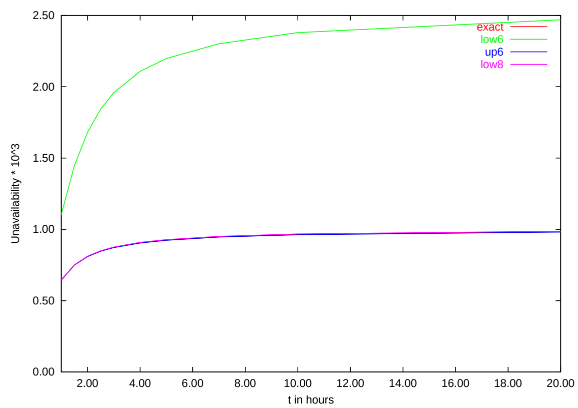


Fig. 3. Bounds and exact results for the unavailability.

it can be observed that the other sets also have greatest and least states. Bounding processes DM^+ and DM^- are show them in Figures 2(a) and 2(b) respectively.

Note that in the processing subsystems, like in almost all models for availability analysis, the most likely states are those in which all components are operational, or one component has failed. Those states are the greatest states in the partition and are used in the upper bounding process. However, in the lower-bounding process, two of those states (MP1P2, SP1P2), are aggregated to states with other failed components. In this case, the accuracy of the aggregation can be improved by placing those two states into singleton sets in the partition, increasing the state space size of the lower-bounding process (shown in Figure 2(c)) from 6 states to 8 states.

To show the quality of the aggregation, the unavailability of the system in the interval $[0, t]$ is computed for t from 0 to 20 hours. The results for the unavailability are shown in Figure 5. Observe that the lower bound aggregates yield an upper bound for the unavailability and the upper bound aggregate yields an lower bound for the unavailability. The 6 state upper bound and the 8 state lower bound aggregate yield excellent results which cannot be distinguished from the exact values in the figure. In fact, the difference between the exact values and the bounds for the unavailability is less than 0.5% over the whole interval. The 6 state lower bound aggregate is less good, it results in an upper bound for the unavailability which is nearly 2.5 times larger than the exact value. Of course, this difference becomes much smaller when the availability is compared, which is much larger than the unavailability, but usually one is interested in the unavailability and results should be compared based on this measure.

6 Conclusions

The paper presents a new preorder relation for discrete time Markov chains with additional rewards, that preserves rewards in the sense that a larger state (process) yields a larger transient or stationary reward than a smaller state (process). The relation of the new preorder to strong stochastic ordering, monotonicity and lumpability is shown. Furthermore it is shown how the new relation can be used to build aggregated processes that bound the results of the original process. Since the preorder can be computed by a stepwise refinement approach it is an example for an order relation tailored to a specific model class and a specific class of results in the sense of [8].

References

[1] P. Buchholz. Exact and ordinary lumpability in finite Markov chains. *Journal of Applied Probability*, 31:59–75, 1994.

[2] P. Buchholz. Efficient computation of equivalent and reduced representations for stochastic automata. *Int. Journ. Computer Systems Science and Engineering*, 15(2):93–103, 2000.

[3] D. J. Daley. Stochastically monotone Markov chains. *Z. Wahrscheinlichkeitstheorie und verw. Geb.*, 10:305–317, 1968.

[4] D. Daly. *Bounded Aggregation Techniques to Solve Large Markov Models*. PhD thesis, Department of Electrical Engineering at the University of Illinois at Urbana Champaign, 2005.

[5] S. Derisavi, H. Hermanns, and W. H. Sanders. Optimal state space lumping in Markov chains. *Information Processing Letters*, 54(2):309–315, 2003.

[6] J. Keilson and A. Kester. Monotone matrices and monotone Markov processes. *Stochastic Processes and their Applications*, 5:231–241, 1977.

[7] J. G. Kemeny and J. L. Snell. *Finite Markov Chains*. Springer, 1976.

[8] A. Müller and D. Stoyan. *Comparison Methods for Stochastic Models and Risks*. Wiley, 2002.

[9] R. R. Muntz and J. Lui. Computing bounds on steady-state availability of repairable computer systems. *Journal of the ACM*, 41(4):676–707, 1994.

[10] W. J. Stewart. *Introduction to the numerical solution of Markov chains*. Princeton University Press, 1994.