Expected Impulse Rewards in Markov Regenerative Stochastic Petri Nets *

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Abstract. Reward structures provide a versatile tool for the definition of performance and dependability measures in stochastic Petri nets. In this paper we derive formulas for the computation of expected reward measures in Markov regenerative stochastic Petri nets, which allow for transitions with non-exponentially distributed firing times. The reward measures may be composed of rate rewards which are obtained in certain markings and of impulse rewards which are obtained when transitions fire. The main result of the paper is the derivation of formulas for the expected impulse reward of transitions with non-exponentially distributed firing times. The analysis is based on the method of supplementary variables. Numerical examples are given for an M/D/1/K queueing system with service breakdowns.

Key words: Markov regenerative stochastic Petri nets, method of supplementary variables, rate and impulse reward measures.

1 Introduction

The inclusion of reward structures in stochastic Petri nets (SPNs) facilitates the specification of a variety of interesting performance and dependability measures. This fact has motivated the introduction of reward structures as an integrated part of modeling in two SPN variants: in stochastic activity networks \cite{19, 20} and in stochastic reward nets \cite{6}. In both cases, rewards are specified at the net level, so that the modeling and the specification of measures is done at the same level. Two types of rewards are typically considered: rate rewards and impulse rewards. Rate rewards are associated with markings of the SPN and are collected

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during the time the SPN resides in the marking. Impulse rewards, on the
other hand, are associated with transition firings and are collected when
a transition fires. In a recently proposed specification of reward structures
[18], impulse rewards may depend on the state in which a transition fires
and on the untimed events which take place immediately after the firing.

If all transition firing times are exponentially distributed, the underly-
ing stochastic process is a Markov chain and formulas are known for
the computation of the expected values of the reward measures (e.g., [6]).
The computation of the distribution of the reward measure is computa-
tionally more expensive, but results are also available (e.g., [16, 17]). If
the firing times of the transitions are generally distributed the underly-
ing stochastic process is not a Markov chain. Under the constraint that in
each marking at most one transition with a generally distributed firing
time is enabled, the underlying process is a Markov regenerative stochas-
tic process [5, 7]. Therefore this class of SPNs is referred to as Markov
regenerative stochastic Petri nets (MRSPNs)

Two approaches have been suggested for the analysis of MRSPNs.
First, an embedded Markov chain can be considered at a suitable selected
subset of time points, an approach taken by most authors. The approach
is suitable for stationary [2, 5, 7, 21] as well as transient analysis [4,
5, 14, 3, 15, 12]. Secondly, the process can be made Markovian by the
inclusion of supplementary variables [8, 9]. This approach has been used
for the stationary analysis of MRSPNs in [11] and has been extended
for the transient analysis in [10, 12]. A comparison of both approaches
can be found in [12]. The aim of the analysis in the reported references
is to compute the vector of state probabilities, either in the transient or
stationary case, and to derive the measures of interest from these state
probabilities. However, if the reward measure contains impulse rewards of
transitions with general firing time distributions, it is no longer possible
to derive the expected reward directly from the state probabilities.

The aim of this paper is therefore to derive formulas for the computa-
tion of expected reward measures in MRSPNs, given that the reward
measures are composed of both rate and impulse rewards. For this purpose
we propose a mathematical framework for the representation of combined
rate and impulse reward measures. This extends the formalisms suggested
in [6, 20]. The analysis is then conducted by the method of supplemen-

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3 This constraint can easily be formulated and checked on a computer. However, the
class of SPNs with an underlying Markov regenerative process is actually larger. It
is also important to consider the firing policy of the transitions. In this paper we
always assume a firing policy known as race with enabling memory. See [3, 21] for a
more detailed discussion.
tary variables. State equations are derived and analyzed which describe the dynamics of the underlying stochastic process. The analysis combines results from [11, 10, 12], and the derivation of this general class of state equations has not yet been published. The general form of the state equations allows the formulation of the main result of this paper: computational formulas for the expected impulse reward of transitions with generally distributed firing times. The result is given as an integral over the value of the supplementary variable. The formula appears natural in the approach of supplementary variables, since the equations are valid for the full set of time points. A comparable result is not known for the approach of the embedded Markov chain.

The paper is organized as follows. In Sec. 2 the considered class of SPNs is introduced. The mathematical formalism for the representation of reward structures is given in Sec. 3. The analysis of the stochastic process is presented in Sec. 4 and formulas for the expected reward are given in Sec. 5. A numerical example is shown in Sec. 6 and conclusions are given in Sec. 7.

2 Markov Regenerative Stochastic Petri Nets

We assume that the reader is familiar with SPNs, and will therefore only briefly discuss the specific class of SPNs considered in this paper. The primitives of the considered of SPNs are: places, transitions, arcs, and indistinguishable tokens. The arcs are divided into input, output, and inhibitor arcs, and the transitions are divided into immediate and timed ones. Furthermore, the timed transitions are divided into those which fire after an exponentially distributed time (referred to as exponential transitions) and those which fire after a generally distributed time (referred to as general transitions).

The set of all transitions is denoted by $T$. $T$ can be partitioned into the set of exponential transitions $T^E$ and the set of general transitions $T^G$. Single transitions are denoted by letters $a, g, h \in T$. The firing time of a transition $g \in T^G$ is specified by the probability distribution function (PDF) $F^g(x)$. We assume that the PDF has no mass at zero \(^4\), (i.e., $F^g(0) = 0$), and use $x_{\text{max}}^g$ to denote the range over which the firing time is defined (i.e., $x_{\text{max}}^g = \min \{x \geq 0 : F^g(x) = 1\}$). If the distribution has infinite support, $x_{\text{max}}^g$ will be interpreted as $x_{\text{max}}^g = \infty$. The firing time can also be represented by the probability density function (pdf) $f^g(x)$ or

\(^4\) This does not affect the use of immediate transitions.
by the *instantaneous rate function* (irf) \( \lambda^g(x) \), defined by:

\[
\lambda^g(x) = \frac{f^g(x)}{1 - F^g(x)}; \quad \text{for } x < x^g_{\text{max}}. \tag{1}
\]

\( \lambda^g(x) \) is undefined for \( x \geq x^g_{\text{max}} \). The firing time may be a mixed random variable containing both a continuous and a discrete part. The discrete part corresponds to discontinuities of \( F^g(x) \). These discontinuities can be represented as Dirac impulses in the pdf and irf (see [13], pp. 341–343 and 372–373 for an informal discussion of generalized functions in the context of random variables). Using this formalism of generalized functions, \( f^g(x) \) and \( \lambda^g(x) \) are always existing. In the special case of a deterministic transition with delay \( \tau \), the PDF and pdf are given by

\[
F^g(x) = \begin{cases} 
0 & x \leq \tau, \\
1 & x > \tau,
\end{cases} \quad f^g(x) = \delta(x - \tau),
\]  

where \( \delta(x - \tau) \) denotes the Dirac unit impulse located at \( x = \tau \). The irf is then given by

\[
\lambda^g(x) = \begin{cases} 
\delta(x - \tau) & x \leq \tau, \\
\text{undefined} & x > \tau.
\end{cases}
\]

In this paper we consider SPNs in which at most one general transition is enabled in each marking and in which the firing policy of each transition is *race with enabling memory* (as defined in [1]). Under this policy, a new delay has to be sampled if a disabled transition becomes enabled again. Since a Markov regenerative stochastic process is underlying an SPN of this class, we refer to these SPNs as *Markov regenerative stochastic Petri nets* (MRSPNs).

In order to illustrate the formalism, an example is given and used throughout the paper. Figure 1 shows an SPN model of an M/D/1/K-queueing system with service breakdowns. The exponential transition \( a_1 \) models arrival of customers and the deterministic transition \( a_2 \) service of customers. Tokens in \( P_2 \) represent customers inside the system. The service facility is operating if a token is in \( P_3 \) and failed if a token is in \( P_4 \). The exponential transitions \( a_3 \) and \( a_4 \) model the failure and repair. The rates \( \lambda, \rho, \) and \( \sigma \) are associated with the exponential transitions \( a_1, a_3, \) and \( a_4 \), respectively, \( a_2 \) has a constant firing delay \( \tau \). The arcs between \( P_2 \) and \( a_3 \) model that a failure is only possible, if the system is utilized. Note that due to the race-enabling memory policy the service time is lost if it is interrupted by a failure.
The tangible markings of an MRSPN constitute the states of the underlying stochastic process. We denote the discrete state space by $\mathcal{S}$ and refer to the single states with integers $i, j \in \mathcal{S}$. It is assumed that the discrete state space is finite. The stochastic process is a tuple of random variables:

$$\left\{ (N(t), X(t)), t \in \mathbb{R}^+_0 \right\},$$

where $N(t)$ gives the tangible marking at time $t$. If in $N(t)$ a general transition $g$ is enabled, $X(t)$ gives the times since the enabling of $g$. Otherwise $X(t)$ is undefined. $N(t)$ is discrete and $X(t)$ is continuous. $X(t)$ is referred to as the supplementary variable.

The example SPN shown in Fig. 1 contains only tangible markings. The discrete state space $\mathcal{S}$ is given by the tuples $(n, m)$, where $n$ gives the number of customers inside the system and $m \in \{o, f\}$ denotes whether the server is in mode operating ($o$) or failed ($f$). The state number is given by $i = n + K \cdot 1_{m=f}$. Figure 2 shows the stochastic process for $K = 5$. Each state transition is labeled with the rate, which is constant ($\lambda$) in case of state transitions caused by exponential transitions and which depends on the value of $X(t)$ in case of state transitions caused by the deterministic transition ($\mu(x)$).

3 Reward Measures

For a formal introduction of reward measures we use a framework similar to that suggested in [20]. This framework has been extended in [18] to allow for more general impulse reward definitions, but to keep the notation concise we will only allow a single impulse reward for each transition on the state-space level. The extension of the results obtained in this paper can easily be extended to include the reward structure in [18]. A reward
structure is given by the vector $r$ and the matrices $C^a$ for each timed transition $a \in T$:

- $r$, where $r_i \in \mathbb{R}$ is a rate reward which is obtained when the SPN is in state $i$,

- $C^a$, where $c_{i,j}^a \in \mathbb{R}$ is an impulse reward which is obtained when the transition $a$ fires in state $i$ and causes a state transition to state $j$.

Furthermore, let $I_i^t$ and $I_{i,a,j}^{t(\tau)}$ be indicator variables:

$$I_i^t = \begin{cases} 1 & N(t) = i \\ 0 & \text{otherwise} \end{cases},$$ (5)

$$I_{i,a,j}^{t(\tau)} = \begin{cases} 1 & a \text{ causes a state transition from } i \text{ to } j \text{ at time } t \\ 0 & \text{otherwise} \end{cases}. \quad (6)$$

These indicator variables can be used to express the reward which is collected in a certain interval of time. Note that the variable $I_i^t$ equals one over all time intervals and that the variable $I_{i,a,j}^{t(\tau)}$ equals one only at single instances of time. In other words, $I_i^t$ represents continuous quantities on each interval and $I_{i,a,j}^{t(\tau)}$ represents discrete quantities located at the instants of transition firings. One can deal with both quantities by integrating $I_i^t$ over a time interval and summing over $I_{i,a,j}^{t(\tau)}$ at the instants of transition firings. The integration and summation can be explicitly expressed. This approach is taken in [6]. Motivated by the use of generalized functions for mixed random variables (as discussed at the beginning of the last section), we represent here the discrete quantities by Dirac impulses. This leads to a compact formalism. The interval-of-time or accumulated reward $Y_{[0,t]}$ gives the rate and impulse reward obtained from 0 to $t$ and is defined as:

$$Y_{[0,t]} = \sum_{i \in S} \int_0^t r_i \cdot I_i^t \, dx + \sum_{a \in T} \sum_{i \in S} \sum_{j \in S} \int_0^t c_{i,j}^a \cdot I_{i,a,j}^{t(\tau)} \cdot \delta(0) \, dx. \quad (7)$$
The Dirac impulse $\delta(0)$ causes a step of the interval-of-time reward at time $t$ of height $c_{i,j}$ if $I_{i,j}(t) = 1$. The time-averaged interval-of-time reward gives the reward obtained per time unit from $0$ to $t$:

$$W_{[t]} = \frac{1}{t} \cdot Y_{[t]}.$$

Both the interval-of-time and the time-averaged interval-of-time reward are random variables. It is possible to define the distribution and expectation of the interval-of-time reward:

$$F_{Y_{[t]}}(y) = \Pr\{Y_{[t]} \leq y\}, \quad E[Y_{[t]}] = \int_{0}^{\infty} y dF_{Y_{[t]}}(y).$$

The expectation exists only in case of absolute convergence of the integral. The distribution and expectation of the time-averaged interval-of-time reward can be defined similarly.

Another interesting quantity is the instant-of-time reward measure. For rate rewards the instant-of-time measure can be defined straightforwardly, but in case of impulse rewards defining a useful interpretation is more involved. Different approaches have been taken to overcome this problem (e.g., [20, 6]). Here we consider the proposal in [6], where the instant-of-time measure is defined as the change of the interval-of-time measure. For rate rewards the resulting measure corresponds to the natural definition of the instant-of-time measure.

We first consider the random variable $W_{[t]+\Delta t]$ which is the difference quotient of the interval-of-time reward:

$$W_{[t]+\Delta t] = \frac{Y_{[t]+\Delta t] - Y_{[t]}}{\Delta t}.$$

The instant-of-time reward $V_t$ could then be defined as the differential quotient $\lim_{\Delta t \to 0} W_{[t]+\Delta t]$. Using the derivation rules for generalized functions the steps of the interval-of-time reward correspond to Dirac impulses in the instant-of-time reward. The value of $V_t$ is infinite at these points and the area is equal to the step height of $Y_{[t]}$. Intuitively, the Dirac impulse describes the change in case of a step. Since $V_t$ contains Dirac impulses it is not a proper random variable. However, we can define its “expectation” as the following limit:

$$E[V] = \lim_{\Delta t \to 0} \int_{0}^{\infty} v dF_{W_{[t]+\Delta t]}(v) = \lim_{\Delta t \to 0} E[W_{[t]+\Delta t}].$$

For the interval-of-time reward and the instant-of-time reward one can define the stationary limits:

$$E[W] = \lim_{t \to \infty} E[W_{[0,t]}], \quad E[V] = \lim_{t \to \infty} E[V_t].$$


These limits not necessarily both exist. Below, an example will be shown where the limit \( E[V] \) does not exist. If \( E[V] \) is existing, it is equal to \( E[W] \). A proof of existence of \( E[W] \) is beyond the scope of this paper.

**Example** An example is investigated in order to illustrate the framework of reward measures. Figure 3 shows an SPN with two deterministic transitions which let the token circulate between the two places. The firing delay of \( a1 \) is equal to 3 and the delay of \( a2 \) equal to 2. Let the state space be \( S = \{0, 1\} \); the state number denotes the number of tokens in \( P2 \). The following reward structure is defined: a rate reward equal to -1 is obtained when \( N(t) = 1 \) and an impulse reward equal to 3 is obtained when \( a2 \) fires. Figure 4 shows the evolution of the instant-of-time reward \( V_t \). \( V_t \) consists of a Dirac impulse when \( a2 \) fires (at multiples of 5). A Dirac impulse is graphically represented by an arrow; the area of the impulse is shown in the circle at the destination of the arrow. \( V_t \) describes the change of the interval-of-time reward. The limit \( E[V] \) is not existing. Figure 5 shows the interval-of-time and time-averaged interval-of-time reward. The time-averaged interval-of-time reward converges to a limit. Note that due to the deterministic nature of the process all reward variables are equal to their expected values. In Fig. 6 the corresponding expected rewards are shown if both transitions have exponentially distributed firing times with unchanged mean firing times.

![Figure 3. SPN with two states.](image3.png)

![Figure 4. Evolution of the instant-of-time reward.](image4.png)
4 Analysis of the Stochastic Process

For the computation of the reward measures of an MRSPN the underlying stochastic process has to be analyzed. We use the method of supplementary variables [8, 9]; based on the state space representation given in Eq. (4) the forward Kolmogorov state equations are formulated. These equations express the state occupancy distribution as function of the time $t$ and the value of the supplementary variable $x$. In Sec. 5 it will be shown, how from these quantities the values of the expected rewards can be derived. In this section we derive the state equations and discuss their analysis. The derivation in this section is a generalization of the material published in [11, 10, 12].

4.1 Transient Analysis

Let $S^E$ contain all states in which only exponential activities are enabled and let $S^G$ contain all states in which general transition $g \in T^G$ is enabled (for each $g \in T^G$ exists a separate set $S^g$). The transient probabilities
are denoted as $\pi_i(t) = \Pr \{N(t) = i\}$. For states with a supplementary variable (i.e., $i \in S^g, g \in T^g$), the age density functions are defined as:

$$\pi_i(t, x) = \frac{d}{dx} \Pr \{N(t) = i, X(t) \leq x\}; \quad (13)$$

$\pi_i(t, x)$ is a probability with respect to $t$ and a (defective) pdf with respect to $x$. $\pi_i(t, x)dx$ represents the probability of being in state $i$ in an infinitesimal environment of $x$ at time $t$. In order to derive the state equations in a generic vector-matrix form the following vectors are defined:

- $\pi^E(t)$, $\pi^0(t)$, and $\pi^0(t, x)$, $g \in T^g$.

All vectors are of dimension $|S|$. $\pi^E(t)$ contains all single probabilities in states of $S^E$, $\pi^0(t)$ and $\pi^0(t, x)$ contain all single probabilities and age densities in states of $S^g$, respectively. The elements that do not correspond to the states indicated by the superscripts are equal to 0. Additionally, the following matrices are defined:

- $Q_{E,g}^E$, $Q_{E,g}^0$, $Q_{g,h}^E$, $Q_{g,h}^0$, and $\Delta_{g,h}$, $g, h \in T^g$.

All matrices are of dimension $|S| \times |S|$. The $Q$-matrices contain the rates of state changes caused by exponential transitions and the $\Delta$-matrices describe branching probabilities after a general transition has fired. The superscripts indicate the subsets of the state space between which the state transitions take place (e.g., $Q_{E,g}^E$ contains all rates of transitions starting in states $i \in S^E$ and leading to states $j \in S^g$). The elements that do not correspond to the states indicated by the superscripts are equal to 0. $Q_{g}^0$ is a special case: it contains all rates of exponential state transitions which can take place during $g$ is enabled and which do not disable $g$, it also contains the negative sum of all outgoing rates in states $i \in S^g$.

In states with a supplementary variable, (i.e., $i \in S^g, g \in T^g$) the forward equations are given by:

$$\pi^g(t + \Delta t, x + \Delta t) = \pi^g(t, x) \cdot (I + Q_{g}^0 \Delta t - \lambda^g(x) \Delta t) + o(\Delta t), \quad (14)$$

for $0 \leq x < x^g_{max}$. $I$ denotes the identity matrix and $o(\Delta t)$ is a function tending to zero more rapidly than $\Delta t$. For states without a supplementary variable (i.e., $i \in S^E$) the forward equations are given by:

$$\pi^E(t + \Delta t) = \pi^E(t) \cdot \left( I + Q_{E,g}^E \Delta t \right)$$

$$+ \sum_{g \in T^g} \int_0^{x^g_{max}} \pi^g(t, x) \cdot \lambda^g(x) \cdot \Delta_{g,E}^E dx \Delta t$$

$$+ \sum_{g \in T^g} \int_0^{x^g_{max}} \pi^g(t, x) \cdot Q_{g,E}^E dx \Delta t + o(\Delta t). \quad (15)$$
Let the vectors $\pi^E_0$, $\pi^s_0$, $g \in T^G$, represent the initial state occupancy distribution. Assuming that no general transition was enabled before $t = 0$ we obtain as initial conditions:

$$\pi^E(0) = \pi^E_0, \quad \pi^s(0) = \pi^s_0, \quad \pi^s(0, x) = \pi^s_0 \cdot \delta(0).$$  \hspace{1cm} (16)

The supplementary variable is set to zero in the instant of enabling of a general transition $g$:

$$\pi^s(t, 0) = \pi^E(t) \cdot Q^E g + \sum_{h \in T^G} \int_0^{x_{\text{max}}^h} \pi^h(t, x) \cdot \chi^h(x) \cdot \Delta^h g \, dx + \sum_{h \in T^G} \int_0^{x_{\text{max}}^h} \pi^h(t, x) \cdot Q^h g \, dx.$$  \hspace{1cm} (17)

Integrating the age densities over $x$ yields the state probabilities:

$$\pi^s(t) = \int_0^\infty \pi^s(t, x) \, dx.$$  \hspace{1cm} (18)

Equations (14–18) describe the dynamic behavior of the process. Following the line presented in [8] these equations can be transformed into a more convenient form. Subtracting $\pi^s(t, x)$ from both sides of Eq. (14), dividing both sides by $\Delta t$, and taking the limit $\Delta t \to 0$ leads to the following system of partial differential equations (PDEs):

$$\frac{\partial}{\partial t} \pi^s(t, x) + \frac{\partial}{\partial x} \pi^s(t, x) = \pi^s(t, x) \cdot Q^s - \pi^s(t, x) \cdot \chi^s(x),$$  \hspace{1cm} (19)

for $0 < x < x_{\text{max}}^s$. In order to simplify the PDE system further, the following quantities are defined:

$$p_i(t, x) = \frac{\pi_i(t, x)}{1 - F^g(x)}; \quad x < x_{\text{max}}^g, i \in S^s; g \in T^G.$$  \hspace{1cm} (20)

Since $p_i(t, x) dx$ can be interpreted as the probability of being in state $i$ in an infinitesimal environment of $x$, given that the firing time has not yet elapsed, $p_i(t, x)$ is referred to as the instantaneous age rate. $p^s(t, x)$ is defined as the vector of all single age rates of $g$. Application of the chain rule for differentiation to the left side of Eq. (19) gives:

$$\frac{\partial}{\partial t} \pi^s(t, x) + \frac{\partial}{\partial x} \pi^s(t, x) = \frac{\partial}{\partial t} p^s(t, x) \cdot (1 - F^g(x)) + \frac{\partial}{\partial x} p^s(t, x) \cdot (1 - F^g(x)) - p^s(t, x) \cdot f^g(x).$$  \hspace{1cm} (21)
Moreover, since \( \pi^g(t, x) \cdot \lambda^g(x) = p^g(t, x) \cdot f^g(x) \), we obtain:

\[
\frac{\partial}{\partial t} p^g(t, x) + \frac{\partial}{\partial x} p^g(t, x) = p^g(t, x) \cdot Q^g.
\]  

(22)

Using the method of characteristics, Eq. (22) can be simplified to a system of ordinary differential equations (ODEs) with solution [10]:

\[
p^g(t_0 + h, x_0 + h) = p^g(t_0, x_0) \cdot e^{Q^g h}, \quad t_0, x_0, h \in \mathbb{R}_0^+.
\]  

(23)

Equation (15) can be transformed into a system of ODEs by subtracting \( \pi^E(t) \) from both sides, dividing both sides by \( \Delta t \), taking the limit \( \Delta t \to 0 \), and substituting the age densities by the age rates:

\[
\frac{d}{dt} \pi^E(t) = \pi^E(t) \cdot Q^{E,E} + \sum_{g \in I^g} \int_0^\infty p^g(t, x) dF^q(x) \cdot \Delta g^E \cdot \sum_{g \in I^g} \pi^g(t) \cdot Q^{g,E}.
\]  

(24)

Substituting the age densities by the age rates in the initial conditions (16), boundary conditions (17), and integral equations (18) yields:

\[
\pi^E(0) = \pi^E_0; \quad \pi^g(0) = \pi^g_0; \quad p^g(0, x) = \pi^g_0 \cdot \delta(x),
\]  

(25)

\[
p^g(t, 0) = \pi^E(t) \cdot Q^{E,g} + \sum_{h \in I^g} \int_0^\infty p^h(t, x) dF^h(x) \cdot \Delta h^g + \sum_{h \in I^g} \pi^h(t) \cdot Q^{h,g}.
\]  

(26)

\[
\pi^g(t) = \int_0^\infty p^g(t, x) \cdot (1 - F^q(x)) \, dx.
\]  

(27)

Equations (23–27) uniquely describe the time-dependent behavior of the stochastic process underlying an MRSPN and constitute transient state equations. The numerical analysis of the transient equations is discussed in [10, 12]: the continuous variables \( t \) and \( x \) can be discretized, the age rates and the state rates can then be computed at the grid points by an iterative scheme.

**Simplifications for Deterministic Firing Times** If a transition \( g \) has a deterministic firing time \( \tau^g \), the integrals in Eqs. (24, 26, 27) simplify to:

\[
\int_0^\infty p^g(t, x) \, dF^q(x) = p^g(t, \tau^g),
\]  

(28)

\[
\int_0^\infty p^g(t, x) \cdot (1 - F^q(x)) \, dx = \int_{\tau^g}^\infty p^g(t, x) \, dx.
\]  

(29)
4.2 Stationary Analysis

The time-averaged limits of the transient state probabilities and age rates are defined as:

\[ \pi_i = \lim_{y \to \infty} \int_0^y \pi_i(t) dt, \quad p_i(x) = \lim_{y \to \infty} \frac{1}{y} \int_0^y p_i(t, x) dt. \]  

(30)

\[ \pi^E, \pi^g, \] and \[ p^g(x) \] are vectors containing the single state probabilities and age rates of states of \( S^E \) and \( S^g \), respectively. Taking the time-averaged limits on both sides of the transient state equations (22, 24 – 27) eliminates the variable \( t \) in these equations. The system of PDEs in Eq. (22) reduces to a system of ODEs:

\[ \frac{d}{dx} p^g(x) = p^g(x) \cdot Q^g. \]  

(31)

The system of ODEs (24) reduces to the following system of balance equations:

\[ 0 = \pi^E \cdot Q^{E,g} + \sum_{g \in T^G} \int_0^\infty p^g(x) dF^g(x) \cdot \Delta^g + \sum_{h \in T^G} \pi^h \cdot Q^{h,g}. \]  

(32)

The boundary conditions (26) simplify to the following equations:

\[ p^g(0) = \pi^E \cdot Q^{E,g} + \sum_{h \in T^G} \int_0^\infty p^h(x) dF^h(x) \cdot \Delta^h + \sum_{h \in T^G} \pi^h \cdot Q^{h,g}. \]  

(33)

Note that the initial conditions (25) are not relevant for the stationary case (a formal characterization of the conditions is beyond the scope of this paper). The integral equations (27) reduce to:

\[ \pi^g = \int_0^\infty p^g(x) \cdot (1 - F^g(x)) dx. \]  

(34)

Finally, an additional normalization condition is given by:

\[ \sum_{i \in S} \pi_i = 1. \]  

(35)

The numerical analysis of the stationary equations is discussed in [11].

The solution of the ODE system (32) is given by the matrix exponential \( p^g(x) = p^g(0) \cdot e^{Q^g x} \). The integrals occurring in Eqs. (31, 33) can then iteratively be computed by a generalized version of Jensen’s method, given that the PDFs \( F^g(x) \) of the firing times can piecewise be represented by exponential polynomials [11, 7]. The remaining equations constitute a linear system of equations in \(|S| \) unknowns (the unknowns are the entries of the vectors \( \pi^E \) and \( p^g(0); \; g \in T^G \)). All state probabilities can be obtained from this solution.
Simplifications for Deterministic Firing Times If a transition $g$ has a deterministic firing time $\tau^g$, the integrals in Eqs. (32, 33, 34) simplify to:

$$\int_0^{\tau^g} p^g(x) \, dF^g(x) = p^g(\tau^g),$$  \hspace{1cm} (36)

$$\int_0^{\tau^g} p^g(x) \cdot (1 - F^g(x)) \, dx = \int_0^{\tau^g} p^g(x) \, dx. \hspace{1cm} (37)$$

5 Computation of Expected Rewards

As discussed in Sec. 3, a reward structure is given by the vector $r$ of rate rewards and by the matrices $C^a$, $a \in T$, of impulse rewards. Let $\pi(t)$ denote the vector of transient state probabilities and let $\pi$ denote the vector of stationary state probabilities. The expected rate reward can be derived from the state probabilities, as can the expected impulse reward corresponding to exponential transitions. However, for the expected impulse reward of general transitions the age densities are required: it can be obtained by integrating the product of the age densities and of the instantaneous rate functions of the firing times over the supplementary variable $x$.

A vector $s$ is defined to contain all constants which have to be multiplied with the state probabilities. These constants are the rate rewards and the impulse rewards of the exponential transitions. The $i$th entry of $s$ is given by:

$$s_i = r_i + \sum_{a \in T^g} \sum_{j \in S} c^g_{ij} \cdot \lambda^a_j \cdot \delta^g_{ij}. \hspace{1cm} (38)$$

$\delta^g_{ij}$ denotes a branching probability (probability that a state transition to state $j$ takes place, given that $a$ fires in state $i$). A vector $d^g$ is defined with all constants needed to compute the expected impulse rewards of the general transition $g$. The $i$th entry of $d^g$ is defined as:

$$d^g_i = \sum_{j \in S} c^g_{ij} \cdot \delta^g_{ij}. \hspace{1cm} (39)$$

The expected instant-of-time reward $V_t$ can now be expressed as:

$$E[V_t] = \pi(t) \cdot s + \sum_{g \in T^g} \int_0^{x^g_{\text{max}}} \pi^g(t, x) \cdot \chi^g(x) \cdot d^g \, dx. \hspace{1cm} (40)$$

The expected impulse reward of a general transition $g$ is obtained by integrating over the product of age densities in which $g$ is enabled and the
instantaneous rate function associated with \( g \) (the impulse reward is thus conditioned on the value of the supplementary variable). Substituting the age density by the age rate according to Eq. (20) modifies the expression:

\[
E[V_t] = \pi(t) \cdot s + \sum_{g \in T^G} \int_0^{\tau_{\text{max}}} p^g(t, x) dF^g(x) \cdot d^g. \tag{41}
\]

Note that the integrals in Eq. (41) have to be computed as an intermediate step during the numerical analysis of the transient state equations (even if one only solves for the state probabilities). Therefore, only minor changes to the solution algorithm are required to compute the expected impulse rewards according to Eq. (41). The expected accumulated reward is given by the integral of Eq. (41):

\[
E[Y_{[0,t]}] = \int_0^t \pi(y) dy \cdot s + \sum_{g \in T^G} \int_0^t \int_0^{\tau_{\text{max}}} p^g(y, x) dF^g(x) dy \cdot d^g, \tag{42}
\]

which can be computed by numerical integration over the values at discrete points.

From Eq. (42) the computation of the expected time-averaged reward \( E[W_{[0,t]}] \) is straightforward. The stationary limit \( E[W] = \lim_{t \to \infty} E[W_{[0,t]}] \) is given by:

\[
E[W] = \pi(s + \sum_{g \in T^G} \int_0^{\infty} p^g(x) dF^g(x) \cdot d^g. \tag{43}
\]

The integrals in this equation also have to be computed during the numerical analysis of the stationary state equations. Hence, as for the transient case, no significant additional computational costs is involved in the computation of expected rewards containing impulse rewards.

**Simplifications for Deterministic Firing Times** If all transitions \( g \in T^G \) have deterministic firing times \( \tau^g \), Eqs. (41, 43) simplify to:

\[
E[V_t] = \pi(t) \cdot s + \sum_{g \in T^G} p^g(t, \tau^g) \cdot d^g, \tag{44}
\]

\[
E[W] = \pi(s + \sum_{g \in T^G} p^g(\tau^g) \cdot d^g. \tag{45}
\]
Special Case: Throughput of a Transition The throughput of a transition is defined as the expected number of firings per time unit and is normally considered in the stationary limit. The throughput is a measure of interest in many performance studies. Since it is a special case of an impulse reward, it is discussed here.

For the throughput of a transition $a$ each firing of $a$ is counted; an impulse reward equal to one is obtained for each firing of $a$. This can be represented by setting all entries of $C^a$ equal to one. The stationary throughput is then given by $S(a) = E[W]$. We define the transient throughput as $S_t(a) = E[V_t]$: this quantity describes the rate of firing of the transition varying over time $t$. In case of a general transition $g \in T^G$ the transient throughput is thus given by:

$$S_t(g) = \int_0^\infty \pi^g(t, x) \cdot \lambda^g(x) \cdot e \, dx$$

$$= \int_0^\infty p^g(t, x) \, dF^g(x) \cdot e$$

(46)

where $e$ denotes a vector containing values equal to one. In the stationary case this simplifies to:

$$S(g) = p^g(0) \cdot \int_0^\infty e^{Q^g x} \, dF^g(x) \cdot e.$$  

(47)

Common knowledge tells that the stationary throughput of a transition $a$ can be computed by summing the probabilities of states in which $a$ is enabled and dividing this sum by the mean firing time $\overline{\lambda^a}$ of $a$. In case of a general transition $g$ (and firing policy race with enabling memory) this result is only valid, if $g$ cannot be disabled before firing; in the following we show that in this case Eq. (48) can be transformed to the simple expression described above. Since $g$ cannot be preempted, the rows of the matrix exponential sum to one, and Eq. (48) can be simplified:

$$S(g) = p^g(0) \cdot \int_0^{\overline{\lambda^a}} e^{Q^g x} \, dF^g(x) \cdot e = p^g(0) \cdot e$$

$$= p^g(0) \cdot \overline{\lambda^a} \cdot e = p^g(0) \cdot \int_0^{\overline{\lambda^a}} e^{Q^g x} \cdot e \cdot (1 - F^g(x)) \, dx \cdot \frac{1}{X^g}$$

$$= \pi^g \cdot e \cdot \frac{1}{X^g}.$$  

(49)

Note that this result is not valid, if $g$ can be disabled before firing. In case the counting of transition firings depends on the state change, however, the expression in (43) can be used in order to to take the branching probabilities into account.
6 Numerical Examples

We have performed numerical experiments for the SPN of the M/D/1/K queueing system with service breakdowns shown in Fig. 1. The parameters of the model have been chosen as follows: \( K = 5, \lambda = 0.5, \tau = 1, \rho = 0.1, \) and \( \sigma = 0.2. \) The initial state of the SPN is as shown in Fig. 1, which corresponds on the state space level to state 0, see Fig. 2. A Mathematica proto-type implementation has been used for the numerical experiments. For all transient results a fixed discretization step-size \( h = 0.01 \) is used (the deterministic service time is thus discretized into 100 steps).

In a first set of experiments the throughput of the exponential transition \( a1 \) (modeling arrivals) and of the deterministic transition \( a2 \) (modeling services) was investigated. Figure 7 shows three curves: the transient throughput \( S_t(a1) \) and \( S_t(a2) \) of \( a1 \) and \( a2, \) respectively, and the stationary throughput of both transitions (the horizontal straight line). It can be seen that the transient throughput of the deterministic transition behaves very irregularly. The transient throughput peaks at multiples of the service time \( \tau = 1, \) but smooths out when \( t \) increases. In the stationary limit the values for \( S_t(a1) \) and \( S_t(a2) \) go to the same limiting value, showing an example of the law of marking flow.

![Figure 7. Transient and stationary throughput of transitions a1 and a2.](image)

A more complicated reward structure is used to compute what can be interpreted as the "gain" of the system. It combines rate rewards and impulse rewards, as follows. A rate reward \(-0.1\) is obtained if at least one token is waiting in the system, which can be interpreted as the cost of an occupied server. Impulse rewards are collected whenever a service
completes (value 1) or when a repair completes (value -5). The negative impulse reward indicates costs while the positive impulse reward indicates benefits. Figure 8 shows the expected instant-of-time gain. The stationary gain is shown by the horizontal straight line in Fig. 8. The instant-of-time gain can be interpreted as time-dependent change of the interval-of-time reward, as explained in Sec. 3. If the actually accumulated reward over a time interval is considered, the curve looks considerably smoother, as can be seen in Fig. 9.

![Graph showing transient and stationary gain](image)

**Fig. 8.** Expected transient and stationary reward "gain".

![Graph showing accumulated gain](image)

**Fig. 9.** Expected accumulated reward "gain".
7 Conclusion

In this paper the analysis of reward measures in Markov regenerative stochastic Petri nets has been investigated. The considered reward measures incorporate both rate and impulse rewards, and results have been derived for the expectation of transient as well as stationary measures. If impulse rewards are considered, the definition and solution of measures becomes more involved. Therefore, a mathematical formalism has been proposed that facilitates convenient definition of reward measures based on rate and impulse rewards, and allows for straightforward incorporation of reward measures in the solution formulas. For the analysis we have successfully adopted the method of supplementary variables. The expected impulse rewards can be expressed by quantities which are required in intermediate steps during the computation of the state probabilities. Therefore no significant additional costs are required.

References


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