STATE-SPACE SUPPORT FOR PATH-BASED REWARD VARIABLES*

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Abstract

Many sophisticated formalisms exist for specifying complex system behaviors, but methods for specifying performance and dependability variables have remained quite primitive. To cope with this problem, modelers often must augment system models with extra state information and event types to support particular variables. This often leads to models that are non-intuitive, and must be changed to support different variables. To address this problem, we extend the array of performance measures that may be derived from a given system model by developing new performance measure specification and model construction techniques. Specifically, we introduce a class of path-based reward variables, and show how various performance measures may be specified using these variables. Path-based reward variables extend the previous work with reward structures to allow rewards to be accumulated based on sequences of states and transitions. To maintain the relevant history, we introduce the concept of a path automaton, whose state transitions are based on the system model state and transitions. Furthermore, we present a new procedure for constructing state spaces and the associated transition rate matrices that support path-based reward variables. Our new procedure takes advantage of the path automaton to allow a single system model to be used as the basis of multiple performance measures that would otherwise require separate models or a single more complicated model.

1. Introduction

Many sophisticated formalisms now exist for specifying complex system behaviors, and many tools exist that can convert a model specified in the formalism to an underlying stochastic process that can be solved. Specification methods for performance and dependability variables, on the other hand, have remained quite primitive by comparison. For example, most stochastic Petri net (SPN) tools require a user to specify performance and dependability variables in terms of a rate defined on the states of the model, and possibly, an impulse defined on each event (e.g., transition in a SPN). To overcome this limitation, one often has to add extra components to a model (e.g. places and transitions if modeling using SPNs) to collect the desired information. These components are not part of the system being modeled, and must change whenever one desires new information from the model.

It is thus often the case that several different models of a system must be built in order to obtain the desired performance measures. Changing the model can be a time-consuming procedure, since it then must be validated to guarantee that it is still an accurate representation of the system under study. We address this problem by extending performance measure specification and state-space construction procedures to allow more flexible use of a given model. This is made possible by 1) extending current reward variable specification methods to include variables that have "state" and can capture behavior related to sequences of events and states, and 2) extending current state-space construction algorithms that build stochastic processes that are tailored to the variable(s) of interest.

The use of performance measures to direct state-space construction is not new, but has been limited to supporting lumping based on symmetries for analytic models. In particular, [15] uses standard rate and impulse-based reward variables to put limits on the lumping that can be achieved because of symmetries in a model, and to support impulses that depend on particular activity completions. Path-based reward variables for impulses have been considered, but only to the extent that their use did not change the state space that is generated. Specifically, [13] considers the use of such variables, but limits their use to impulses on sequences of instantaneous events in order not to change the set of (stable) states that is generated.

Our work extends previous work in two important ways. First, we provide support for a more general class of reward variables for a given system model. In particular, we support the definition of measures that depend on sequences of states and events that may occur. Examples of variables whose specification is facilitated by these methods include computations of probabilities of occurrence of particular recovery actions, which have multiple steps, and computation of measures related to consecutive cell loss in ATM networks, among others. We do this by introducing the "path automaton," a finite automaton that can be

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used to define rewards on sequences or sets of sequences of system model states and/or transitions (both timed and untimed). By building the required memory into the performance measure specification, we simultaneously accomplish two goals: we make more flexible the specification of complex performance and dependability variables, and we avoid the need to develop multiple system models. This approach offers the advantage of a single, smaller, system model that is easier to construct and validate. Multiple performance measures defined on multiple path automata may then be defined relative to the single system model.

Second, we provide procedures for automatic construction of a state space that supports the specified variables from the definition of the system model, path automata, and reward structures. Note that the choice of variables as well as system model determine the state space that is generated, and different variables result in different size state spaces for the same system model. In addition, these state generation procedures include automatic support for state-space truncation for the case of performance measures defined over intervals terminated upon satisfaction of a condition on the system model, such as entrance to a particular state or the occurrence of a sequence of states and/or transitions. This is made possible by the use of path automaton "final states," which are interpreted by the state-space construction procedure as indicating that the model state reached upon entry to the final state should not be explored any further.

The remainder of the paper is organized as follows. The next section describes a model specification formalism that we use to simplify the exposition of the ideas in this paper. Section 3 introduces the new concept of a path-based reward variable, and Section 4 shows how various performance measures may be specified using path-based reward variables. Then, in Section 5, we present new procedures for automatically generating a state space that supports multiple path-based reward variables, and summarize the relevant numerical solution techniques. Section 6 gives an example model and some results on the variation of state space size for different performance measures.

2. Model specification

In this section we review a simple model description formalism, defined in [10]. Our objective in introducing this formalism is to simplify the presentation of our ideas for variable specification and state-space construction, which are independent of the details of the actual modeling language. While this formalism is well suited to our purpose, it is a low-level formalism, and leads to verbose and sometimes unwieldy model definitions. However, many different high-level modeling languages can be mapped to this formalism. We demonstrate one such mapping through our use of a stochastic activity network in Section 6.

Definition 1 *A* model *is a five-tuple* $(S, E, \varepsilon, \lambda, \tau)$ *where*

- S *is a set of state variables* $\{s_1, s_2, \ldots, s_n\}$ *that take values in* $I\!N$*, the set of nonnegative integers. The state of the model is defined as a mapping* $\mu : S \rightarrow I\!N$*, where for all* $s \in S$*,* $\mu(s)$ *is the value of state variable* s*. Let* $M = \{\mu \mid \mu : S \rightarrow I\!N\}$ *be the set of all such mappings.*

- E *is the set of events that may occur.*

- $\varepsilon : E \times M \rightarrow \{0, 1\}$ *is the event enabling function. For each* $e \in E$ *and* $\mu \in M$*,* $\varepsilon(e, \mu) = 1$ *if event* e *may occur when the current state of the model is* μ*, and zero otherwise.*

- $\lambda : E \times M \rightarrow (0, \infty)$ *is the transition rate function. For each event* e *and state* μ *such that* $\varepsilon(e, \mu) = 1$*, event* e *occurs with rate* $\lambda(e, \mu)$ *while in state* μ*.*

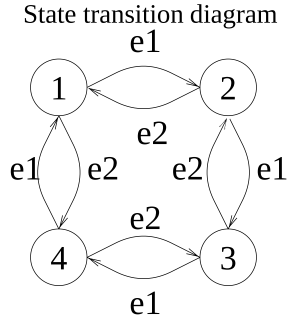
- $\tau : E \times M \rightarrow M$ *is the state transition function. For each* $e \in E$ *and* $\mu \in M$*,* $\tau(e, \mu) = \mu'$*, the new state of the model that is reached when* e *occurs in* μ*.*

We now present a simple example to illustrate this modeling formalism. We will also use the same example in Section 5 to illustrate our state-space construction methods. Consider the state transition diagram in Figure 1. This system has a single state variable and two events, each of which is enabled in each state. Thus $S = \{s_1\}$ and $E = \{e_1, e_2\}$. The state variable takes values in the set $\{1, 2, 3, 4\}$, leading to state mappings $\mu_1 = (s_1, 1)$, $\mu_2 = (s_1, 2)$, $\mu_3 = (s_1, 3)$ and $\mu_4 = (s_1, 4)$. Since each event is enabled in each state, $\varepsilon(e, \mu) = 1$ for all $(e, \mu) \in E \times M$. We define the rate of occurrence of e_1 in any of the four states to be ϕ, and the rate of e_2 to be θ. The definition of the model is given in the tables of Figure 1.

A model executes as follows. Starting in a state, μ, the enabled events, $E(\mu) = \{e \mid \varepsilon(e, \mu) = 1\}$, compete to cause the next state transition. Each event occurs in μ at rate $\lambda(e, \mu)$. If event e occurs in μ, the next state is given by $\tau(e, \mu)$. The probability distribution over the set of states that may be reached in one transition from μ is determined in the usual way from the relative magnitudes of the event occurrence rates. In the example model of Figure 1, the mean holding time in each state is $\frac{1}{\phi+\theta}$ and the probability of a clockwise move via e_1 is $\frac{\phi}{\phi+\theta}$.

Models are created to answer questions about a system. We give such questions the generic name "performance measures," which we use in a broad sense, encompassing all the standard performance, dependability, and performability measures that have been defined in the literature. As pointed out in [6], directly formulating performance measures defined at the system level in terms of a stochastic process representation of a model is not always feasible, due to the complexity of the mapping. Therefore, it is necessary (as well as desirable) to define performance measures at a high level. The concept of a "reward structure" has been established as a useful technique for specifying performance measures on models [5, 1, 16]. Defined at the model level, a reward structure associates reward accumulation rates with model states and impulse rewards with model events. Performance measures are then defined in terms of a reward

State transition diagram



Transition function and rate

State	Event	Next State	Rate
1	e_1	2	ϕ
	e_2	4	θ
2	e_1	3	ϕ
	e_2	1	θ
3	e_1	4	ϕ
	e_2	2	θ
4	e_1	1	ϕ
	e_2	3	θ

State variables

Identifier	Description
s_1	model state

Event set

Identifier	Description	Enabling Condition
e_1	clockwise	always enabled
e_2	counter clockwise	always enabled

Figure 1. Example model

structure and a variable specification, which together define a random variable called a *reward variable.* Common examples of such reward variables are the instantaneous reward at time t and the reward accumulated over the interval $[t, t+l]$.

3. Path-based reward variables

We wish to evaluate a performance measure that is based on a sequence of model states and events. We call such a sequence a "path," and such measures "path-based performance measures." Formally,

Definition 2 *A* path, $(\mu_1, e_1)(\mu_2, e_2), \ldots, (\mu_n, e_n)$*, is a sequence of ordered pairs where* μ_i *is a model state and* e_i *is a model event.*

We say a path is *initialized* when the model enters the first state in the path. A path *completes* when the last pair in the sequence is satisfied. A path is *aborted* if the sequence is violated, for example if e_3 occurs in μ_2 instead of e_2. Definition 2 identifies a particular path, and there are many such paths in any given model. Sometimes the level of detail supported in this definition of path is not needed. For example, suppose that we are only interested in a sequence of events e_1, e_2, e_3, without regard to which states are visited. If there are many possible states that can be visited during this sequence of events, then many different paths must be specified. For situations like this, we need a convenient formalism for describing sets of paths.

To facilitate path-set specification, we introduce the "path automaton." The inputs to the automaton are a model event and the model state in which it occurred. For example, (e, μ), indicates that event e occurred in model state μ. The automaton state transition function defines the next state in terms of the current state and the input pair. Formally,

Definition 3 *A* path automaton *defined on a model,* $(S, E, \varepsilon, \lambda, \tau)$*, is a four-tuple,* (Σ, F, X, δ)*, where*

- Σ *is the nonempty set of internal states;*
- F *is a (possibly empty) set of final states;*
- $X = E \times M$ *is the set of inputs; and*
- $\delta : \Sigma \times X \to \Sigma \cup F$ *is the state transition function, where for any internal state* $\sigma \in \Sigma$ *and input pair* $x \in X$*,* $\delta(\sigma, x)$ *identifies the next state.*

The path automaton executes as follows. Starting from an initial state $\sigma_0 \in \Sigma$, input pairs from the model are read, one for each state transition of the model. For each input pair $x \in X$, the state transition function $\delta(\sigma, x)$ identifies the next automaton state $\sigma' \in \Sigma \cup F$. If $\sigma' \in F$ then the path automaton halts. We say a path is *distinguished* by an automaton if completion of the path leaves the automaton in a final state.

We propose to evaluate path-based performance measures using path automata, together with "path-based reward structures."

Definition 4 *A* path-based reward structure *defined on path automaton* (Σ, F, X, δ) *is a pair of functions*

- $\mathcal{C} : \Sigma \times X \to I\!R$*, the* impulse reward function*, where for all internal states* $\sigma \in \Sigma$ *and input pairs* $x \in X$*,* $\mathcal{C}(\sigma, x)$ *is the impulse reward earned when the path automaton is in state* σ *and receives input pair* x*.*
- $\mathcal{R} : \Sigma \times M \to I\!R$*, the* rate reward function*, where for all internal states* $\sigma \in \Sigma$ *and model states* $\mu \in M$*,* $\mathcal{R}(\sigma, \mu)$ *is the rate at which reward is earned when the path automaton is in state* σ *and the model is in state* μ*.*

The path-based reward structure is a generalization of the standard (state-based) reward structure, since depending on the internal state of the path automaton, different rate rewards can be assigned to the same model state and different impulse rewards to the same model event. This is not possible with standard reward structures. On the other hand, any standard reward structure is easily represented using a path-based reward structure where the path automaton has only one state.

Reward variables can be constructed from the path-based reward structure and several random variables defined on the evolution of the model and the path automaton. The following random variables serve as components from which we may construct a broad array of performance measures:

- $I^t_{(\sigma,e,\mu)}$ is an indicator random variable representing the event in which at time t, the path automaton is in state σ and the last input pair was (e, μ);
- $J^{[t,t+l]}_{(\sigma,e,\mu)}$ is a random variable representing the total time during the interval $[t, t+l]$ that the automaton is in state σ and the last input pair was (e, μ); and
- $N^{[t,t+l]}_{(\sigma,e,\mu)}$ is an indicator random variable representing the number of times within the interval $[t, t+l]$ that the automaton is in state σ and receives input pair (e, μ).

Following the approach in [15], we can now define the reward variables we need for evaluating the various performance measures.

$$V_t = \sum_{(\sigma,x)\in\Sigma\times X} (\mathcal{C}(\sigma, x) + \mathcal{R}(\sigma, \mu)) \cdot I^t_{(\sigma,x)}$$

is the instantaneous reward at time t. Note that in the usual case $\mathcal{C}$ would be zero for this type of variable. Otherwise the interpretation is that the impulse associated with the most recent event that occurred prior to t is accumulated along with the rate reward corresponding to the current automaton and model state pair.

$$Y_{[t,t+l]} = \sum_{(\sigma,x)\in\Sigma\times X} \mathcal{C}(\sigma, x) \cdot N^{[t,t+l]}_{(\sigma,x)} + \mathcal{R}(\sigma, \mu) \cdot J^{[t,t+l]}_{(\sigma,\mu)}$$

is the reward accumulated over the interval $[t, t+l]$.

$$W_{[t,t+l]} = \frac{Y_{[t,t+l]}}{l}$$

is the time-averaged accumulated reward over the interval $[t, t+l]$.

So far we have considered reward variables that give us access to the value of the reward structure at fixed times or over intervals defined by fixed times. Some path-based performance measures, such as time to completion, call for an interval that is based on the random time at which some event occurs. In the literature on stochastic processes such

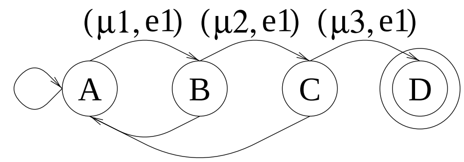


Figure 2. Path automaton for probability of completion of $(\mu_1, e_1)(\mu_2, e_1)(\mu_3, e_1)$

random times are called *stopping times.* Performance measures based on stopping times are easily handled by our formalism for path-based reward variables. We define the random variable T_F to be the instant that the path automaton enters F, the set of final states. Now we can define V_{T_F} as the instantaneous reward immediately following entry to F, and $Y_{[t,T_F]}$ as the reward accumulated from time t until entry to F.

In the next section, we show how path-based reward variables can be used to obtain various performance measures.

4. Example performance measures

Given a model and a path, there are many different questions one might ask. First, we may want to know the probability of traversing the path. Given that the path is traversed, how long does it take? How many times was the path completed in some interval? What is the chance of finding the model in the middle of traversing the path, at some arbitrary time point? What is the total time spent traversing the path within some interval? In this section we show how all of these questions may be answered using path-based reward variables.

The model described in Figure 1 will be used to demonstrate the use of path-based reward variables. First we identify a path. Consider the following sequence of events: starting from state 1, the process visits states 2, 3, and 4 in exactly that order, with no other excursions. The path $(\mu_1, e_1)(\mu_2, e_1)(\mu_3, e_3)$ captures this sequence of events.

To compute the probability of traversing the path before time t, we use the path automaton shown in Figure 2, where a path completion causes a transition to a final state.

The state transition diagram in Figure 2 maps to the formal tuple-notation as follows. States of the automaton appear in the diagram as labeled circles. If there are final states, they are denoted by two concentric circles. Each arc between two states is labeled with the input that causes the transition represented by the arc. Unlabeled transitions are treated as "else" conditions. When we do not label an arc, we mean that the transition is taken if the input does not match a labeled arc.

The probability of traversing the path before time t is then the probability of finding the path automaton in its final state at time t. Thus we use the reward structure

$$\mathcal{C}(\sigma, x) \quad = \quad 0$$

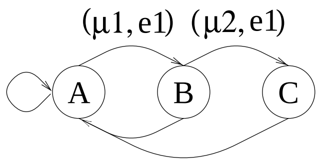


Figure 3. Path automaton for number of completions of $(\mu_1, e_1)(\mu_2, e_1)(\mu_3, e_1)$

$$\mathcal{R}(\sigma, \mu) = \begin{cases} 1 & \text{if } \sigma = D \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and evaluate $E[V_t]$, the expected value of V_t. Since $V_t = 1$ if the path completed and $V_t = 0$ otherwise, $E[V_t]$ is the probability that the path completes before t.

To compute the time to completion of the considered path, we use the path automaton in Figure 2, and the reward structure

$$\mathcal{C}(\sigma, x) = 0$$
$$\mathcal{R}(\sigma, \mu) = \begin{cases} 1 & \text{if } \sigma \neq D \\ 0 & \text{otherwise.} \end{cases}$$

The time to completion is the reward variable $Y_{[0,T_F]}$ where, for this automaton, $F = \{D\}$.

To count the number of completions of this path, we need to assign an impulse reward of 1 to (μ_3, e_1), but only if this event is preceded by $(\mu_1, e_1)(\mu_2, e_1)$. In addition, we do not want path completion to be a final state, since this performance measure is defined over multiple completions. The path automaton in Figure 3 is suitable for this performance measure. The reward structure for counting the number of completions of the path is

$$\mathcal{C}(\sigma, x) = \begin{cases} 1 & \text{if } \sigma = C \text{ and } x = (\mu_3, e_1) \\ 0 & \text{otherwise} \end{cases}$$
$$\mathcal{R}(\sigma, \mu) = 0. \tag{2}$$

An impulse reward of 1 is assigned to the transition from path automaton state C to A if this transition is caused by event ϕ occurring in model state C. The number of completions of the path within the interval $[t, t+l]$ is $Y_{[t,t+l]}$.

The probability of finding the model on the path at time t and the time spent traversing the path within some interval are closely related; they have the same reward structure. Each performance measure can be computed by using the path automaton in Figure 3 and the reward structure

$$\mathcal{C}(\sigma, x) = 0$$
$$\mathcal{R}(\sigma, \mu) = \begin{cases} 1 & \text{if } \sigma = A \text{ and } \mu = \mu_1 \\ 1 & \text{if } \sigma \in \{B, C\} \\ 0 & \text{otherwise.} \end{cases}$$

The expected value of V_t gives us the desired result. To get the total time spent in states on the path in the interval $[t, t+l]$ we evaluate $Y_{[t,t+l]}$.

In most cases we want to be able to obtain as many different performance measures as possible from a single state space. Our method supports multiple performance measures on a single state space, but they must be "compatible," in a way which we will now describe precisely. A single state space can only support (one or more) fixed times, or one (random) stopping time. For this reason, if a stopping time, T_F, is used, we restrict the set of reward variables defined on a single state space such that each reward variable in the set is defined in terms of T_F. Thus two performance measures are *compatible* if they both refer to fixed times or they both refer to the same stopping time. Suppose that we define two path-based reward variables, v_1 and v_2, on a model, and that v_1 is defined on a path automaton that has a final state. For example, v_1 might measure the time to first completion of some path, p_1, while v_2 counts the number of completions of a different path, p_2. Clearly, v_1 and v_2 are most likely defined on different path automata. Yet, as stated previously, if a stopping time is present, all variables must be defined relative to that time in order to be supported by the same state space. Thus v_2 is subordinate to v_1, in the sense that v_2, if it is to be supported by the same state space as v_1, is understood to measure the number of completions of p_2 prior to the time of the first completion of p_1. For example, if we desire results for constant time t and stopping times T_{F_1} and T_{F_2}, three different state spaces are needed. Another ramification of this restriction is that a state space can only support one path automaton that has a nonempty set of final states.

This section has demonstrated how a variety of performance measures can be specified using path-based reward variables. Specific examples related to performance and dependability are given in Section 6. In the next section, we discuss the problem of constructing a state space that supports path-based reward variables.

5. State-space support

We have now presented path-based reward variables and demonstrated how to use them to derive various performance measures. In this section we present a method for constructing state spaces that support these variables.

The first step is to provide a precise definition of a state. We wish to allow multiple path-based reward variables to be associated with a given model. In general, this means that there will be multiple path automata and multiple reward structures to manage. Suppose that there are n different reward variables defined on a model. Each reward variable comprises a path automaton and a path-based reward structure. We index the path automaton and reward structure definitions by $i = 1, 2, \ldots, n$, so that, for example, δ_i is the state transition function for the i-th path automata. Thus we are led to the following definition of a state.

Definition 5 *For a model and a set of n path-based reward variables, a* state *is a four-tuple* $(\sigma[], \mu, c[], r[])$ *where*

- $\sigma[]$ *is an array of path automaton states, where* $\sigma[i]$ *is the internal state of the i-th path automaton;*

U : unexplored states
$E(s)$: events that may occur in s;
1. Initial state $s_0 = (\sigma_0[], \mu_0, \mathbf{0}, r_0[])$
2. $U = \{s_0\}$
3. $S = \{s_0\}$
4. while $U \neq \emptyset$
5.   choose $s \in U$
6. $U = U - \{s\}$
7. $E(s) = \{e \in E \mid \varepsilon(e, s.\mu) = 1\}$
8.   for each $e \in E(s)$
9. $\mu' = \tau(e, s.\mu)$
10.   for $i = 1$ to n
11. $\sigma'[i] = \delta_i(s.\sigma[i], s.\mu, e)$
12. $c[i] = \mathcal{C}_i(s.\sigma[i], s.\mu, e)$
13. $r[i] = \mathcal{R}_i(\sigma'[i], \mu')$
14. $s' = (\sigma'[], \mu', c[], r[])$
15.   if $s' \notin S$
16. $S = S \cup \{s'\}$
17.   if $\sigma'[1] \notin F$
18. $U = U \cup \{s'\}$
19.   add arc from s to s' with rate $\lambda(e, s.\mu)$

Figure 4. Procedure for constructing a state space that supports multiple path-based reward variables

- *μ is the state of the model;*
- *$c[]$ is an array of impulse rewards for this state, where $c[i]$ is the impulse reward for the i-th reward variable; and*
- *$r[]$ is an array of rate rewards for this state, where $r[i]$ is the rate reward for the i-th reward variable.*

Note that $r[]$, the array of rate rewards, is determined by σ and μ, the automaton and model states, so it does not add to the state space. However, it is convenient for our presentation to include the complete reward structure in the notion of state. In an implementation, some additional flexibility can be achieved by defining rate rewards separately from the states. As stated at the end of Section 4, the set of path automata supported by a single state space can only include one automaton where F is nonempty. We use the convention that if there is a path automaton with $F \neq \emptyset$, it takes the first position in the array of automaton definitions.

Figure 4 shows a procedure for constructing state spaces that support multiple path-based reward variables. In line 1, the initial state s_0 is identified by the initial state of each path automaton, the initial model state, and the initial reward structure values for each reward variable. By convention, the initial impulse rewards are zero, which is indicated by $\mathbf{0}$. In lines 2 and 3 we initialize the set of unexplored

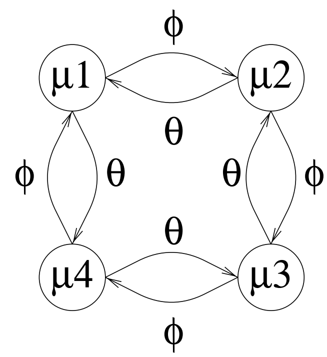


Figure 5. Markov process state transition diagram for simple model

states and the set of visited states. At line 4, starting from the initial state, a breadth-first search of the state space is conducted.

First, we choose a member, s, of the set of unexplored states (line 5). The next step (line 7) is to find the set of events, $E(s)$, that may occur in the current state. For each event the next model state, μ', is found (lines 8–9) by evaluating the model state transition function τ. At this point we have all the information needed to compute the reward structure for the new state. For each reward variable, we compute (lines 10–13) the next automaton state, the impulse reward for e occurring in μ, and the rate reward for the new automaton state, model state pair. These elements form the new state, s', constructed in line 14.

Then (line 15) we check to see if we already visited this state. If not, s' is a new state and we add it to the set of states (line 16). As long as the first path automaton in the array did not enter a final state as a result of the event that just occurred, we also add the new state to the set of unexplored states (line 18). We do not add the state to the unexplored set if it corresponds to a final automaton state, because in this case we want the state to be an absorbing state in this state space, even if it would not be an absorbing state in the model. We do this in order to support evaluation of reward variables defined in terms of stopping times. We will discuss this further at the end of this section. Finally, in line 19 we add a transition from the current state s to the new state s'. The process just described repeats until there are no remaining unexplored states.

To demonstrate the construction procedure, we use the simple example of Figure 1. The Markov process constructed directly from the model in Figure 1, without accounting for any performance measures, is shown in Figure 5. We now construct state spaces for two path-based performance measures, one of which is defined on an interval determined by a stopping time. The first example is the probability of path completion. To support this variable, we use the path automaton of Figure 2 with the reward structure in (1). The state space is shown in Figure 6.

As a second example, we consider the number of completions within an interval $[t, t + l]$. For this performance

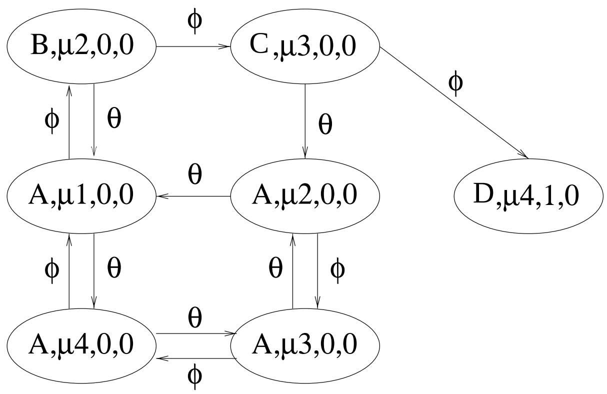


Figure 6. State-space supporting probability of completion of path $(\mu_1, e_1), (\mu_2, e_1), (\mu_3, e_1)$ **by time** t

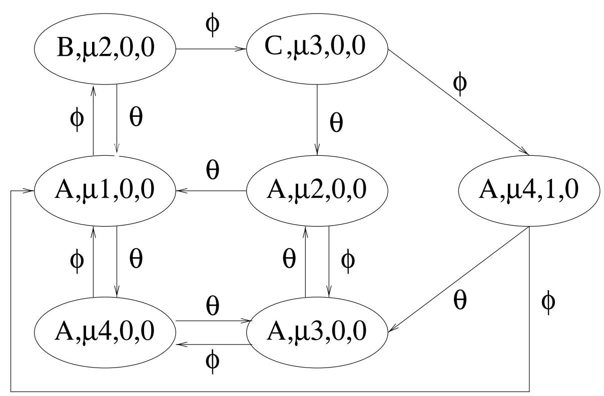


Figure 7. State-space supporting number of completions of path $(\mu_1, e_1), (\mu_2, e_1), (\mu_3, e_1)$

measure, we use the path automaton in Figure 3 and the reward structure in (2). The constructed state space is shown in Figure 7.

After the state space has been constructed, the model needs to be solved for the performance measures of interest. Table 1 summarizes examples of solution procedures that can be used to obtain the reward variables defined in Section 3. In the table, π is the vector of steady-state occupancy probabilities and SOR stands for Successive Over-Relaxation. The distribution of the instantaneous reward variable V_t is completely determined by the state occupancy probabilities at time t, which may be computed using uniformization for time t or by standard Gauss-Seidel or SOR for the limiting case $V_{t\to\infty}$. For $Y_{[t,t+l]}$, the expected value is easily obtained using uniformization. Techniques for calculating the distribution of $Y_{[t,t+l]}$ are much more sophisticated but available (see, for example, [17, 4, 11, 3, 12, 8]).

To evaluate $Y_{[0,T_F]}$, the methods described in [5, 2, 9, 18] can be used to compute the distribution of the reward accumulated until absorption. For $Y_{[t,T_F]}$ we need the state

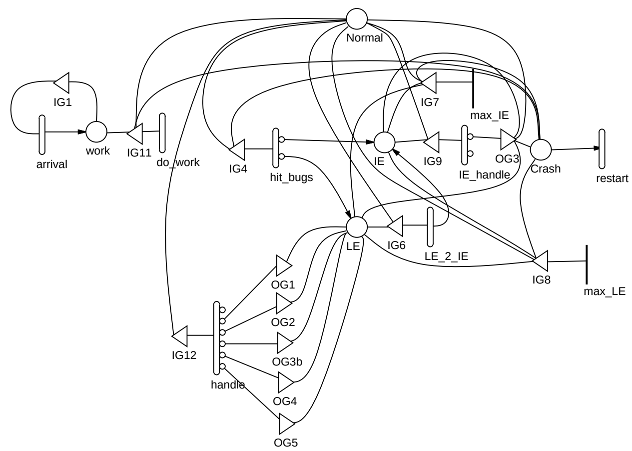


Figure 8. SAN model of fault-tolerant computer system

occupancy probabilities at time t, which we then use as the initial state distribution for the methods used for $Y_{[0,T_F]}$. The time-averaged accumulated reward, $W_{[t,t+l]}$, is easily derived from Y.

6. Fault-tolerant computing example and results

The size of the state space that is needed to support a path-based reward variable clearly depends on the underlying model and the nature of the path automaton. In this section we introduce a larger model and investigate the variation in the size of the state space required for several path-based reward variables.

As mentioned in Section 2, the modeling formalism introduced there and used to develop the variable specification and state-space construction procedures is not intended to be used for large models. For the example model in this section, we use a stochastic activity network (SAN) [7] to represent the system. Each place in the SAN is a state variable in the formalism of Section 2. The possible stable markings of the SAN correspond to state variable mappings in the formalism. We now define the mapping between events in our formalism and events in a SAN. The completion of a timed activity, a case selection, and a sequence of instantaneous activity completions, if any instantaneous activities are enabled, is mapped to an event in our formalism. Using SAN terminology, we map each possible "stable step" [14] to an event in the formalism. The state transition function is determined by the execution rules of the SAN, as are the event enabling function and the event rate function.

We model a computer system designed to function in the presence of software faults. A stochastic activity network model of the system is shown in Figure 8. The system has two modes of operation: normal and diagnostic. In normal mode, there is one token in place *Normal*, and in diagnostic

Table 1. Methods for evaluating the reward variables

Reward Variable	Solution Method	Obtainable Information
V_t	Uniformization	Distribution
$V_{t\rightarrow\infty}$	Gauss-Seidel, SOR	Distribution
$Y_{[t,t+l]}$	Uniformization	Expected value
	Special methods (e.g. [3, 4, 8, 11, 12, 17])	Distribution
$Y_{[t,T_F]}$	Linear system (e.g. [2, 5, 9, 18])	Moments
$Y_{[t\rightarrow\infty,t+l]}$	$Y_{[0,l]}$ starting with π	Distribution

mode, there are zero tokens in *Normal*. Jobs arrive to be processed according to a Poisson process defined by timed activity *arrival*. The finite buffer capacity is modeled by input gate *IG1*. When there are jobs to be processed, the system stays in normal mode. Jobs are processed at a rate defined by timed activity *do_work*. During operation in normal mode various program bugs may be encountered, at a rate governed by timed activity *hit_bugs*. The result of exercising a bug may be an immediately effective error (case 1), or a latent error. An example of an immediately effective error is an address violation; corruption of a data structure is an example of a latent error.

Immediately effective errors are handled by special error handling routines that attempt to recover from the error. The error handling operation is modeled by timed activity *IE_handle* and its two cases. If the error is mishandled (case 1), the system crashes. Furthermore, coincident errors can not be handled, so if an immediately effective error arrives before the last one is handled, the system crashes. This transition is modeled by instantaneous activity *max_IE* and input gate *IG7*. Latent errors do not generate any immediate problems, but the number of latent errors in the system (number of tokens in place *LE*) affects the rate at which immediately effective errors occur. This is modeled by the marking dependent rate of timed activity *LE_2_IE*. We assume that the maximum number of latent errors that can actually be tolerated is five, so if the number of tokens in place *LE* exceeds five, input gate *IG8* enables instantaneous activity *max_LE* to model the resulting system crash.

Upon completing the available workload, the system runs a diagnostic program to check for errors. Input gate *IG11* controls the switch to diagnostic mode. When the completion of timed activity *do_work* leaves zero tokens in place *work*, *IG11* removes the token from place *Normal*, thus initiating diagnostic mode. This program is capable of detecting and removing latent errors. The latent error detection and removal process is modeled by timed activity *handle* and its six cases. The outcome of the diagnostic program depends on the number of latent errors in the system. This dependency is modeled by the six marking-dependent case probabilities defined for timed activity *handle*.

Finally, when the system crashes, it automatically begins a restart operation, which is essentially a reboot. The time it takes to restart the system is modeled by timed activity *restart*. Upon restarting, the system is as good as new.

There are many different performance measures that can be defined for this system. Standard non-path-based measures include the distribution of the number of jobs waiting to be serviced, the availability of the system, and the fraction of time spent running diagnostics.

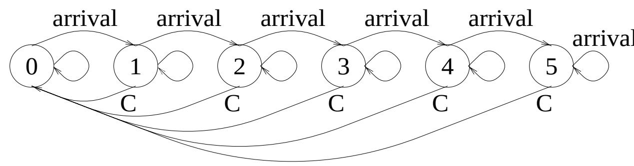


Figure 9. Path automaton for detecting runs of length $k \geq 5$

Several path-based performance measures are also interesting. A path automaton is well-suited to the problem of evaluating the number of times the buffer blocks as a result of a given number of consecutive job arrivals before a job completion. We refer to a sequence of k consecutive job arrivals as a *run of length* k. Figure 9 shows the state transition diagram of a path automaton that can be used to count the number of runs of length $k \geq 5$. Starting from automaton state zero, the automaton records arrivals by increasing its state for each arrival. If a job completion occurs, the automaton resets. If any other event occurs, the automaton remains in its current state.

Using the path automaton of Figure 9, we define three reward variables. The first (3) is simply the number of times the system blocks, which translates to the number of times a job arrives and causes the buffer to be full.

$$\begin{aligned} \mathcal{C}(\sigma,x) &= \begin{cases} 1 & \mu(work) = W_{\max} - 1 \text{ and} \\ & e = \text{arrival} \\ 0 & \text{otherwise.} \end{cases} \\ \mathcal{R}(\sigma,\mu) &= 0 \end{aligned} \tag{3}$$

The second (4) reward variable has an impulse of 1 on the event in which the automaton reaches state five, which corresponds to a run length of at least 5.

$$\begin{aligned} \mathcal{C}(\sigma,x) &= \begin{cases} 1 & \text{if } \sigma = 4 \text{ and } e = \text{arrival} \\ 0 & \text{otherwise.} \end{cases} \\ \mathcal{R}(\sigma,\mu) &= 0 \end{aligned} \tag{4}$$

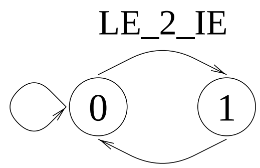


Figure 10. Path automaton for number of crashes caused by a latent error becoming effective and being mishandled

Finally, the third variable (5) records an impulse of 1 each time the buffer becomes full while the automaton is in state 5, which corresponds to the system blocking on a run of at least 5.

$$\mathcal{C}(\sigma, x) = \begin{cases} 1 & \text{if } \sigma = 5 \text{ and} \\ & \mu(work) = W_{\max} - 1 \text{ and} \\ & e = \text{arrival} \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$
$$\mathcal{R}(\sigma, \mu) = 0$$

Blocking events are primarily of interest for performance reasons. From a dependability standpoint, we are more interested in the processes that govern the transition from normal operation to a system crash. Understanding the dominant failure mechanism in a system is important in deciding where to spend time and money to improve the dependability. An example of a path-based dependability measure is the number of crashes in an interval that result from a latent error becoming effective and then being mishandled by the error handling routines. This event is modeled by completion of timed activity *LE_2_IE* followed by completion of *IE_handle* and selection of case 1. The path automaton that captures this sequence of events is shown in Figure 10.

The automaton spends most of its time in state 0, but when timed activity *LE_2_IE* completes, the automaton moves to state 1. Upon the next event, the automaton returns to state 0. The reward structure

$$\mathcal{C}(\sigma, x) = \begin{cases} 1 & \text{if } \sigma = 1 \text{ and} \\ & \text{e = } \textit{handle\_IE}\text{, case one} \\ 0 & \text{otherwise.} \end{cases}$$
$$\mathcal{R}(\sigma, \mu) = 0$$

assigns an impulse reward of 1 to the event in which *LE_2_IE* completes and is immediately followed by *IE_handle*, case 1.

Table 2 shows the sizes of the state spaces required to support the example performance measures. To see how the state space grows with increasing path length, we constructed state spaces for run lengths of 5, 10, 20, 30 and 40. Figure 11 shows the state space growth corresponding to increasing run length.

Table 2. Required state space sizes for various performance measures

Description	State Space Size
Number of model states	1586
Number of crashes via the sequence LE_2_IE, IE_handle, case 1	1952
Number of runs in the workload process that are ≥ 2 and Number of overflows preceded by a run ≥ 2	3826
Number of runs in the workload process that are ≥ 3 and Number of overflows preceded by a run ≥ 3	4567
Number of overflows and Number of overflows preceded by a run ≥ 4	4567
Number of runs in the workload process that are ≥ 4 and Number of overflows preceded by a run ≥ 4	5295

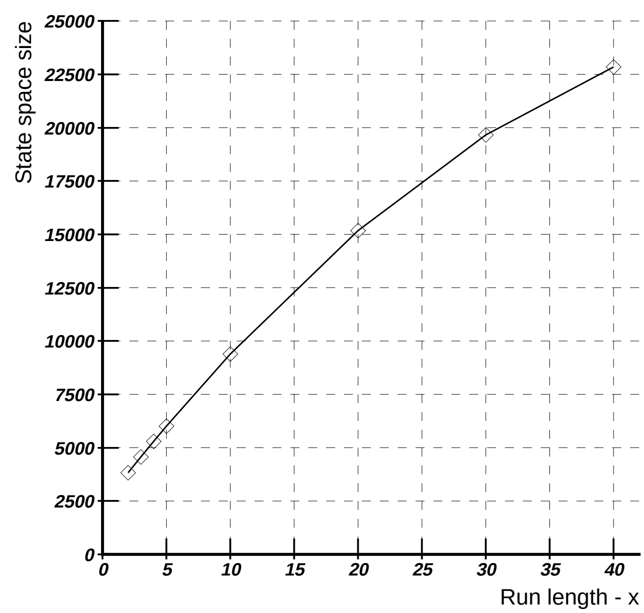


Figure 11. State space size versus arrival run length

7. Conclusion

This paper presents a new performance/dependability measure specification technique and state space construction procedure that, together, solve the problem of supporting a broad array of performance measures from a given system model. Using this new approach, performance measures based on model states, model events, or sequences of (model state, model event) pairs can be supported by a single system model. Furthermore, the performance measures may be evaluated in steady-state, at an instant of time, or over an interval of time defined by fixed endpoints or in terms of a (random) stopping time.

More specifically, we have shown how to specify path-based performance measures using the notion of a path automaton. With the current model state and model event as the basis for path automaton state transitions, we can use the path automaton to specify in a compact way variables that depend on particular sets of sequences of model states and events, as well as those that can be represented by standard reward variables. In addition, we have developed a state generation procedure that makes use of one or more path automata and the system model to generate a state space that is tailored to the variables of interest. In this way, a single system model can support many different performance/dependability variables, and drive the generation of many different state-level models, depending on the measure(s) of interest.

Finally, we have illustrated the use of these measure specification and state generation techniques on several small examples, to illustrate the versatility of the approach, and on a more realistic fault-tolerant computing example, to illustrate the variation in state-space size that can be expected for different measures. These results show the diversity of measures that can be supported by a single model.

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