Combining Adaptive and Standard Uniformization*

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1 Abstract

Adaptive uniformization (AU) has recently been proposed as a method to compute transient measures in continuous-time Markov chains and has been shown to be especially attractive for solving large and stiff dependability models.

The major advantage of AU is that it requires at most as many iterations as standard uniformization (SU), and often far fewer, thus resulting in substantial computational savings. However, this computational gain can be offset by the need to compute more complex "jump probabilities" in AU, whose computation is more expensive than computing Poisson probabilities in SU. In particular, it can be shown that AU is computationally superior to SU if and only if the considered time instant is less than some threshold time value.

To overcome this computational drawback, we combine AU and SU such that AU is used over the start of the time interval of interest, while SU is applied to the rest of the time interval. We show that combined AU/SU can be implemented in such a way that the combination introduces only minor computational overhead, the number of iterations required is almost as low as AU, and the cost of computing the jump probabilities is as low as SU.

To demonstrate the benefits of combined AU/SU, we apply it to a machine-repairman model, using a version of combined AU/SU implemented in the performance and dependability evaluation software package *UltraSAN*.

2 Background

Let $Y = \{Y(s); s \geq 0\}$ be a CTMC defined on the state space $S = \{0, 1, 2, \ldots\}$, and let $Q = (q(i,j)), i, j \in S$, be the infinitesimal generator matrix of Y, where, for notational convenience, we define $q_i = -q(i,i)$ for $i \in S$. Furthermore, let $\underline{\pi}(0) = (\pi_0(0), \pi_1(0), \ldots)$ denote the initial probability distribution over S. The objective is to obtain the transient-state probability vector $\underline{\pi}(t)$, with $\pi_i(t) = \text{Prob}\{Y(t) = i\}$, $i \in S$.

In SU, the CTMC is decomposed into a discrete-time Markov chain (DTMC) and a Poisson process. Let $\lambda \geq \max\{q_i\}, i \in S$, be the so-called *uniformization rate*. Then, the transition matrix $P = I + (1/\lambda)Q$ defines a DTMC $X = \{X_n, n = 0, 1, \ldots\}$, and $\underline{\pi}(t)$ is then equal to

$$\underline{\pi}(t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \underline{\pi}(n), \qquad (1)$$

where $\underline{\pi}(n) = \underline{\pi}(n-1)P$, for $n = 1, 2, \ldots$, and $\underline{\pi}(n = 0) = \underline{\pi}(t = 0)$. In an actual implementation of SU, the infinite sum is truncated (from above and below), such that the resulting error is less than some pre-defined ϵ_s.

In AU [4, 5], the uniformization rate can change with the number of jumps, or current epoch, considered and depends on the set of states the DTMC X can be in at the epoch. Formally, the set of *active states* at epoch $n, n = 0, 1, \ldots$, of X is defined as $A_n \subseteq S$, where

$$A_n = \{i \in S \mid \underline{\pi}_i(n) > 0\}. \qquad (2)$$

Then the "adapted" uniformization rates ("AU rates") are defined as $\lambda_n \geq \max\{q_i \mid i \in A_n\}$, for $n = 0, 1, \ldots$. Next, the "adapted" transition matrices are given by

$$P_n = I + \frac{1}{\lambda_n} Q_n, \quad n = 0, 1, \ldots, \qquad (3)$$

where $Q_n = (q_n(i,j)), i, j \in S$, is the generator matrix Q restricted to the active states at each epoch, that is,

$$q_n(i,j) = \begin{cases} q(i,j) & \text{if } i \in A_n \\ 0 & \text{otherwise.} \end{cases}$$

With $U_n(t)$ defined as the probability of n jumps in time t in the birth process formed by the AU rates $\lambda_0, \lambda_1, \ldots$, then the transient-state probability vector can be defined as [5]

$$\underline{\pi}(t) = \underline{\pi}(0) \sum_{n=0}^{\infty} U_n(t) \prod_{i=0}^{n-1} P_i = \sum_{n=0}^{\infty} U_n(t) \underline{\pi}(n), \qquad (4)$$

where $\underline{\pi}(n) = \underline{\pi}(n-1)P_{n-1}, n = 1, 2, \ldots$, with $\underline{\pi}(n = 0) = \underline{\pi}(t = 0)$. As with SU, the sum is truncated, such that the resulting error is less than ϵ_a.

By using a more general (than Poisson) birth process to describe the jump probabilities, it is possible to make "larger" jumps with AU than SU. Thus, for identical $\epsilon_s = \epsilon_a$, we have that AU requires less iterations, potentially resulting in a reduction of computation time. On the other hand, we know from [1] that the computation of $U_n(t)$ in AU is more costly than the computation of the Poisson probabilities in SU, resulting in a large computation time for AU if

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t is large. The next section introduces the combined AU/SU method, which will be shown to be almost as efficient as AU for small t and outperform both AU and SU for large t.

3 Combined AU/SU

The objective of the combined approach is to find a method that has jumps as long as those in AU and for which the effort of computing the jump probabilities is as low as in SU. To do this, we observe that in many cases the AU rates in AU no longer change after some number of iterations m (i.e., $\lambda_i = \lambda$ for $i \geq m$), since all states will typically eventually become, and remain, active. We speak in this case of a *converged* uniformization rate. When the rate converges, AU jumps are of equal length as those in SU from epoch m on, and there is no further benefit in applying AU.

The basic idea behind combining AU and SU is that computation of $\underline{\pi}(t)$ can be done in two steps—the computation of $\underline{\pi}^c(t_s)$ given $\underline{\pi}(0)$ as the initial distribution and the computation of $\underline{\pi}^c(t)$ starting from $\underline{\pi}^c(t_s)$ at time t_s. This decomposition allows us to apply AU for part of the computation, where it is profitable to do so, and apply SU for the remaining part, where it is superior to AU. t_s will be called the "switching time," and we would like to choose t_s such that we can truncate the sum in AU when the rate converges.

We then have the following algorithm for combined AU/SU:

Algorithm 1 Combined AU/SU

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Choose δ, ε_a and ε_s
Find t_s such that
     Σ_{n=0}^{m} U_n(t_s) = δ;
Compute π(t_s) using AU with π(0) as
     initial state distribution, i.e.,
     π(t_s) = Σ_{n=0}^{m} U_n(t_s) π(n) with
     π(n=0) = π(t=0), and
     δ such that the error bound is ε_a;
Compute π(t) using SU starting from π(t_s),
     i.e., π(t) = Σ_{n=L_s}^{R_s} e^{-λ(t-t_s)} ((t-t_s)^n / n!) π(n)
     with π(n=0) = π^c(t_s) and
     L_s and R_s such that the error
     bound is ε_s;
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To determine the switching time t_s efficiently, we use an approximation method described by Yoon and Shanthikumar ([7], approximation P_4). The basic idea is that, if q is the uniformization rate and $P(q) = I + (1/q)Q$, then

$$\lim_{q \to \infty} P^{qt}(q) = P(t), \tag{5}$$

with $P_{i,j}(t) = Pr\{B(t) = j | B(0) = i\}$. This approximation method is specifically suitable for the problem at hand since the state space of the birth process $B(t)$ is usually quite small, typically no larger than 100 states. Therefore, fill-in resulting from matrix-matrix products is not a problem, and $P^{2^k}(q)$ can thus be computed by k matrix-matrix multiplications: $P^2(q) = P(q)P(q)$, $P^4(q) = P^2(q)P^2(q)$, $P^8(q) = P^4(q)P^4(q)$, etc. Having computed $P^l(q)$ for some l, we find that the probability that more than m jumps take place is given by $P^l_{(0,m+1)}(q)$, with a corresponding time point $t = l \times (1/q)$ ($l = qt$ in (5)). To find the switching time, we thus need to find a value of l such that $P^l_{(0,m+1)}(q) \approx \delta$.[1]

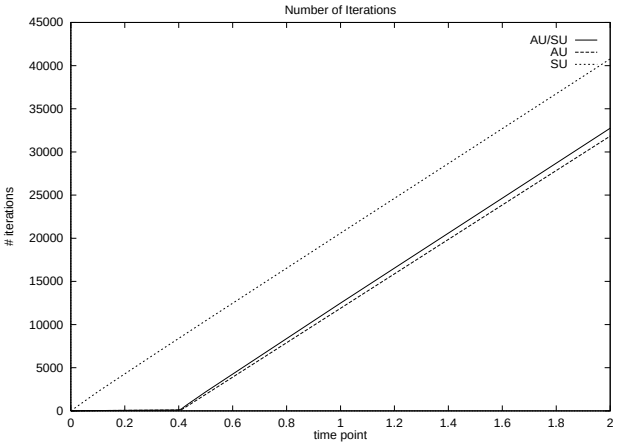

Figure 1: Comparison of the number of iterations in uniformization methods.

4 Practical Considerations and Example

To illustrate the practical considerations and benefits of combined AU/SU, we apply the method to an extended machine-repairman model (EMR model), also used in [1, 2] as an example of a stiff (i.e., computationally intensive) Markov model. The results we present have been obtained from the implementation of combined AU/SU in the performance and dependability evaluation software package *UltraSAN* [3].

We consider an EMR model with K components, all with the same failure rate, ρ. There are two classes of failure, *hard* and *soft*, where the probability of a failure being soft is c and of its being hard is $1 - c$. Repair is delayed until only r components remain functional. The failure and repair times are all negative exponentially distributed random variables with rates ρ (for failures), μ (for repairs of hard-failed components), and ν (for repairs of soft-failed components). Each component has independent repair capability, and repair continues until all units are operational. For highly reliable systems, usually $\mu \gg \rho$ and $\nu \gg \rho$.

The basic model we consider has $K = 200$ components, and repair is initiated when only $r = 100$ components are still functioning. The other parameters for the basic model are $\rho = 1$, $\nu = 100$, $\mu = 80$, and $c = 0.5$. We are interested in the reliability of the system, with the system considered to have failed if all of the components have failed. This reliability model has 25,150 states.

Figure 1 shows the number of iterations required with AU, SU, and combined AU/SU to obtain the specified error bound of $\epsilon_{AU/SU} = 10^{-6}$. In combined AU/SU, we count both the iterations in the AU part and in the SU part.

[1] For details about error bound results (the total error in combined AU/SU can be shown to be bounded by $\epsilon_a + \epsilon_s - \epsilon_a \epsilon_s$), and the interplay between the parameters δ, ϵ_a and ϵ_s, we refer to [6].

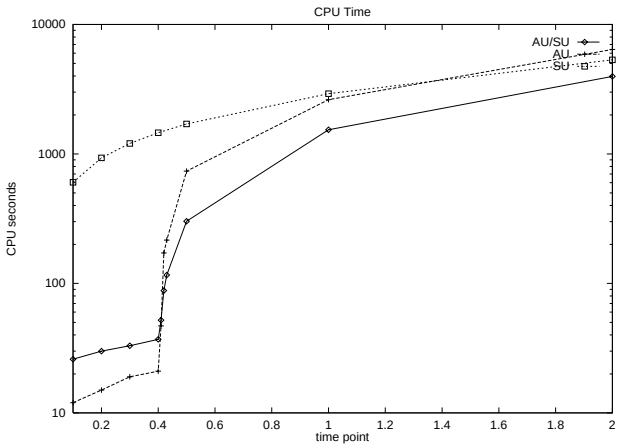

Figure 2: Comparison of the computation time in uniformization methods.

From Figure 1, it follows that the number of iterations in combined AU/SU is close to the number in AU, while the number in SU is considerably higher. The switching time is $t_s = 0.406$, and the number of iterations in combined AU/SU therefore starts to rise when $t > 0.406$, that is, when the SU uniformization rate comes into play.

The number of iterations saved with combined AU/SU gives an idea of the maximally obtainable computational gain with the method, but to determine which method is computationally superior, the computational overhead in combined AU/SU should be considered. Figure 2 plots therefore the used CPU time for the example. This and all runs were made on an HP 715/64 workstation, using implementations of AU, SU, and combined AU/SU made in *UltraSAN*. The times reported are iteration times and do not include the time to read in the matrix from disk, since that is independent of the method used. The computational overhead in combined AU/SU consists mainly of determining the switching time, but it also includes the computation of the jump probabilities in the AU part of combined AU/SU.

The main conclusion to be drawn from Figure 2 (as well as from other figures in [6]) is that, for the considered models, combined AU/SU is superior to SU for all investigated time values. For time points smaller than t_s, AU is superior, but for larger time points, AU becomes far too expensive, making combined AU/SU the preferred method for practical problems.

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